Dynamics and Patterns in Sheared Granular Fluid : Order Parameter Description and Bifurcation Scenario

NDAMS Workshop @ YITP

1st November 2011

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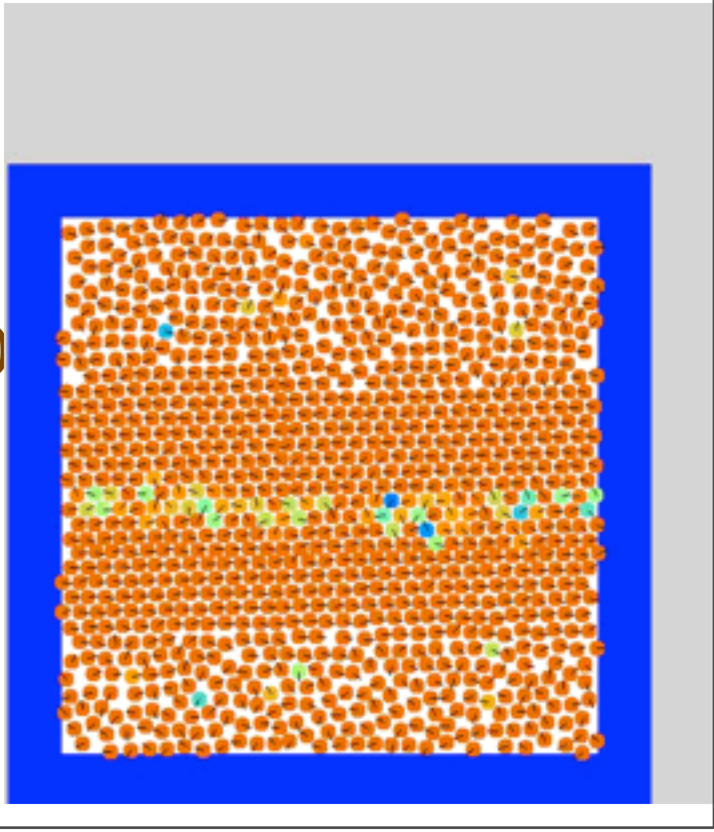
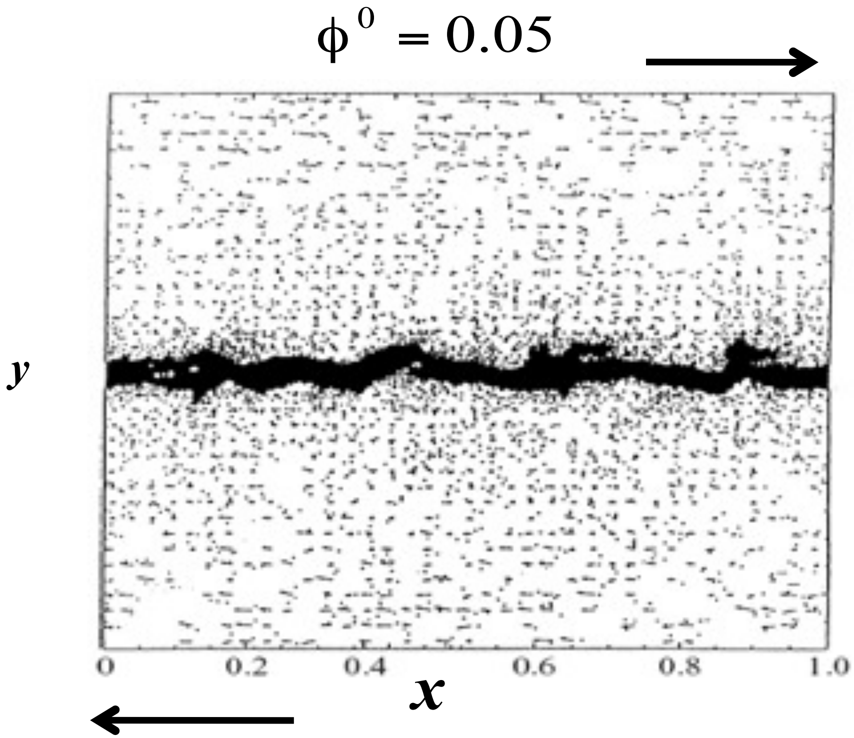
Outline of Talk

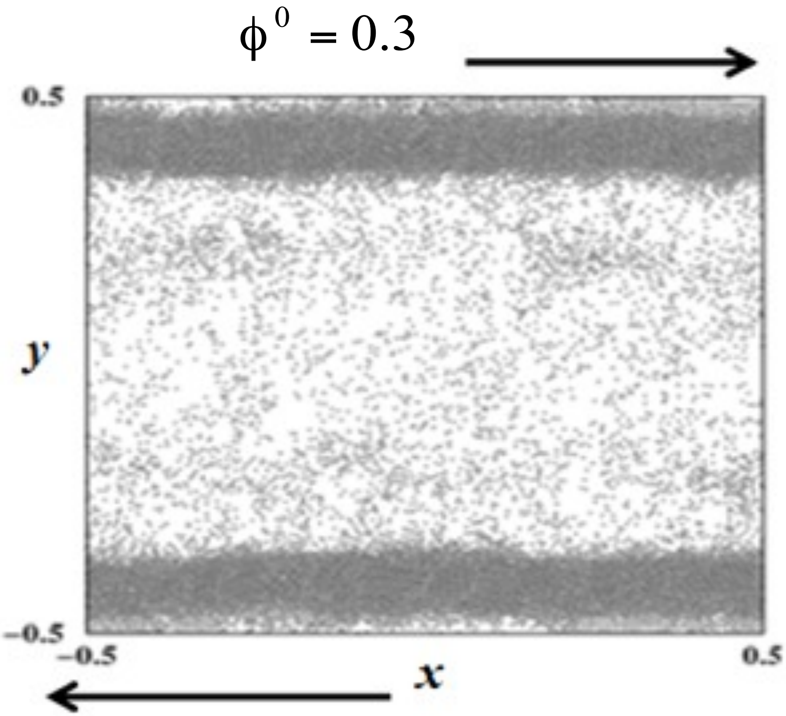
- **Shear-banding phenomena**
- **Gradient Banding and Patterns in 2D-gPCF**
- **Vorticity Banding in 3D-gPCF**
- **Theory for Mode Interactions**
- **Spatially Modulated Patterns (CGLE)**
- **Summary**
- **Possible Connection: Saturn's Ring**



Gradient Banding in 2D-gPCF



Tan & Goldhirsch 1997



Alam 2003

$$\frac{\partial}{\partial x} = 0, \quad \frac{\partial}{\partial y} \neq 0,$$

Order-parameter description of shear-banding?

Shukla & Alam (2009, 2011)

Saitoh &Hayakawa (2011)

Granular Hydrodynamic Equations

(Savage, Jenkins, Goldhirsch, ...)

Balance Equations

Mass $$\frac{D\rho}{Dt} = -\rho\nabla.u$$

Momentum $$\rho\frac{Du}{Dt} = -\nabla.\sum$$

Pseudo Thermal Energy

$$\frac{\dim}{2}\rho\frac{DT}{Dt} = -\nabla.q - \sum : \nabla u - D$$

ϕ : Volume fraction of particles

T : Granular temperature

u : Streamwise velocity

v : Normal velocity

$$\rho = \rho_p\phi$$

Navier-Stokes Order Constitutive Model

Stress $$\sum = (p - \zeta(\nabla.u))I - 2\mu S$$

$$S = \frac{1}{2}(\nabla u + \nabla u^T) - \frac{1}{\dim}(\nabla.u)I$$

Flux of pseudo-thermal energy

$$q = -\kappa\nabla T$$

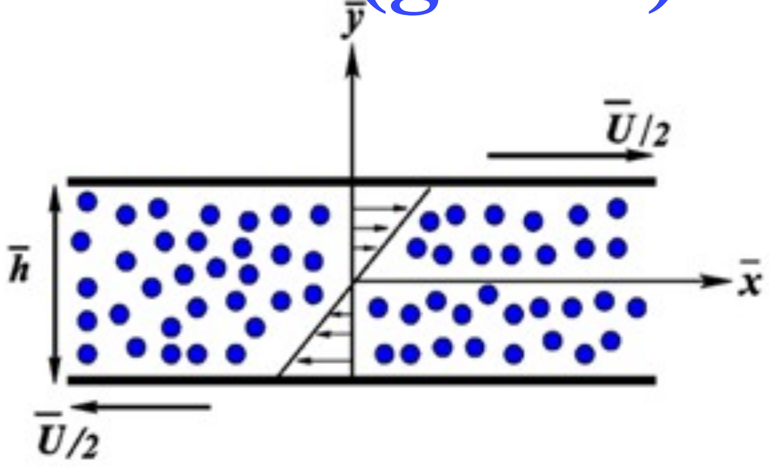
dim: Dimension of system

κ: Thermal Conductivity

μ: Shear Viscosity

D: Sink of granular energy

Plane Couette Flow (gPCF)



d : Particle diameter

Reference Length	$\bar{h}$
Reference velocity	$\bar{U}$
Reference Time	$\bar{h}/\bar{U}$

- **Base Flow Assumption:** Steady, Fully developed.
- **Boundary condition:** No Slip, Zero heat flux.

$$\frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) = 0$$

$$\frac{\partial p}{\partial y} = 0$$

$$\frac{\partial}{\partial y}\left(\kappa \frac{\partial T}{\partial y}\right) + \mu \left(\frac{\partial u}{\partial y}\right)^2 - D = 0$$

$\Rightarrow$ **Uniform Shear Solution**

$$\boxed{\begin{aligned} &\phi^0 = const. \;\; T^0 = const. \\ &u^0(y) = y \end{aligned}}$$

Control paramters

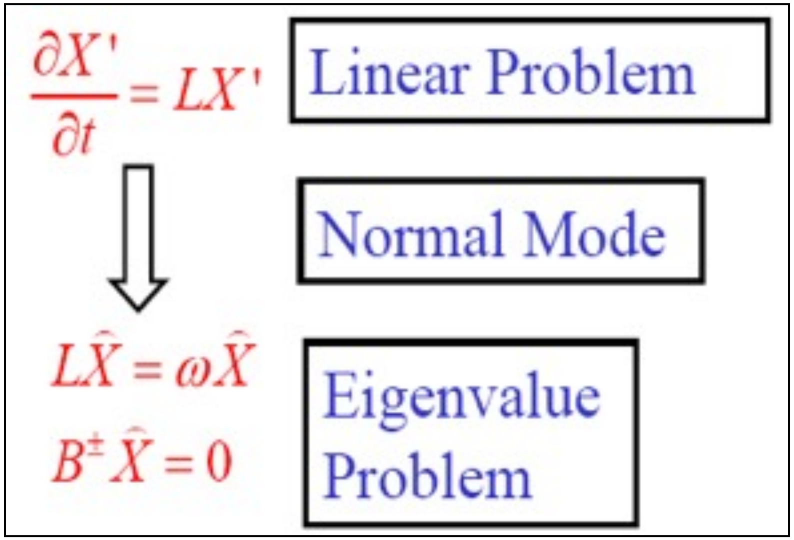
- $H = \bar{h}/d$ Couette Gap
- e Restitution Coeff.
- ϕ^0 Volume fraction or mean density

Linear Stability

$$\boxed{\begin{array}{l}\phi^0 = const.\quad T^0 = const.\\ u^0(y) = y\end{array}} \; +$$

$$= X_{total}$$

If the disturbances are infinitesimal ‘Nonlinear terms’ of the disturbance eqns. can be ‘neglected’.



$$\frac{\partial X'}{\partial t} = LX'$$

Linear Problem

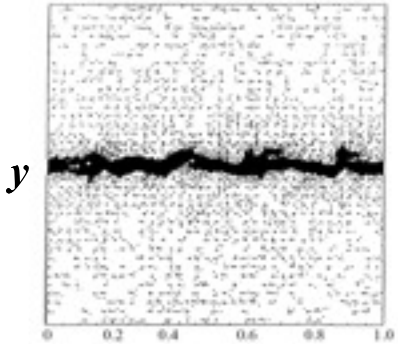
$$\Downarrow$$

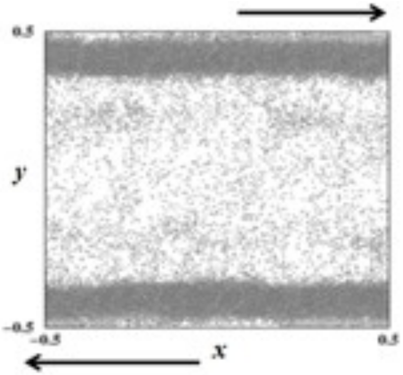
Normal Mode

$$L\hat{X} = \omega\hat{X}$$

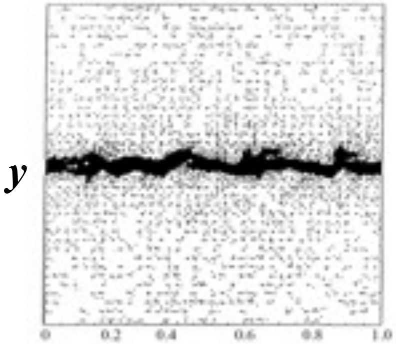
$$B^{\pm}\hat{X} = 0$$

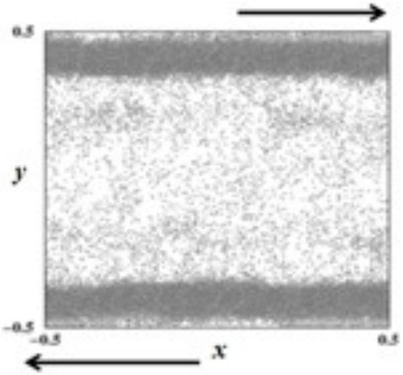
Eigenvalue Problem





Can ‘Linear Stability Analysis’ able to predict ‘Shearbanding’ in Granular Couette flow as observed in Particle Simulations?



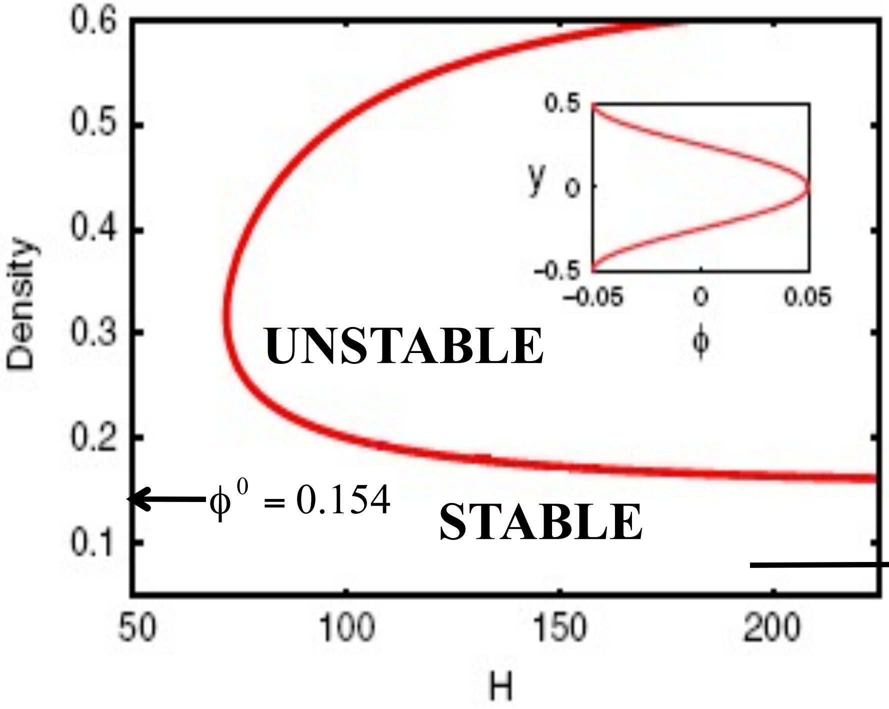


Can ‘Linear Stability Analysis’ able to predict ‘Shearbanding’ in Granular Couette flow as observed in Particle Simulations?

Not for all flow regime

Linear Theory

Shukla & Alam 2009, PRL, 103, 068001



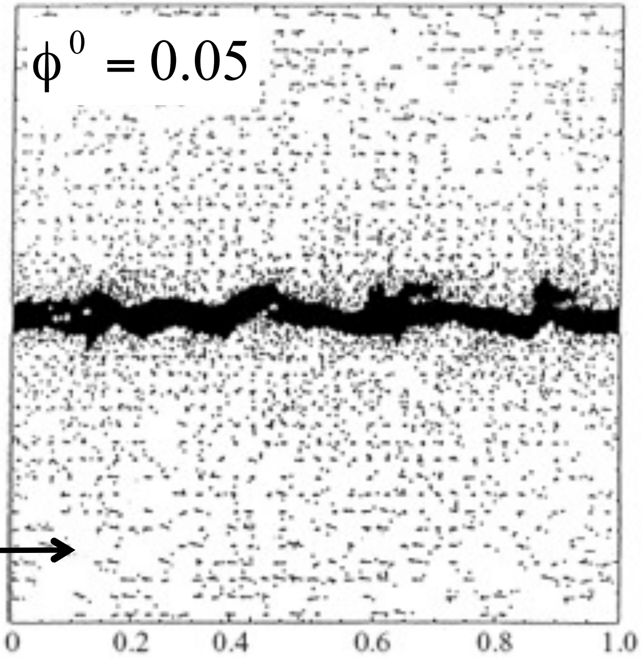
Flow remains ‘uniform’ in dilute limit

Density segregated solutions are not possible in dilute limit

?

Particle Simulation

Tan & Goldhirsch 1997 Phys. Fluids, 9



Flow is ‘non-uniform’ in dilute limit

Density Segregated solutions are possible in dilute limit

We must look beyond Linear Stability

Nonlinear Stability Analysis: Center Manifold Reduction

(Carr 1981; Shukla & Alam, PRL 2009)

Dynamics close to critical situation is dominated by finitely many "critical" modes.

Z : complex amplitude of finite amplitude perturbation

Disturbance

Critical Mode

Non-Critical Mode

$$X' = \phi + \psi$$

$$\phi = ZX^{[1;1]} + \tilde{Z}\tilde{X}^{[1;1]}$$

Amplitude

Linear Eigenvector

$$\boxed{\begin{array}{lcl} \left(\frac{\partial}{\partial t} - L\right)\phi = N_2 + N_3 & \longrightarrow & \left(\frac{\partial}{\partial t} - \omega\right)ZX_{11} = N_2 + N_3 \\ \left(\frac{\partial}{\partial t} - L\right)\psi = N_2 + N_3 & & \left(\frac{\partial}{\partial t} - L\right)\psi = N_2 + N_3 \end{array}}$$

Taking the inner product of slow mode equation with adjoint eigenfunction of the linear problem and separating the like-power terms in amplitude, we get Landau equation

$$\boxed{c^{(0)} = a^{(0)} + ib^{(0)} = \omega}$$

$$\left(\frac{\partial}{\partial t} - \omega\right)ZX_{11} = N_2 + N_3 \longrightarrow \frac{dZ}{dt} = c^{(0)}Z + c^{(2)}Z|Z|^2 + c^{(4)}Z|Z|^4 + ..$$

First Landau Coefficient

Second Landau Coefficient

$$\boxed{c^{(2)} = a^{(2)} + ib^{(2)}}$$

$$\boxed{c^{(4)} = a^{(4)} + ib^{(4)}}$$

Cont…

Adjoint

Distortion of mean flow

Second harmonic

$$c^{(2)} = \frac{\langle Y,\ N_2(X^{[0:2]}, X^{[1:1]}) + N_2(X^{[2:2]}, \tilde{X}^{[1:1]}) + N_3(X^{[1:1]}, X^{[1:1]}, \tilde{X}^{[1:1]})}{\langle Y, X^{[1:1]} \rangle}$$

$$\left(\frac{\partial}{\partial t} - L\right)\psi = \text{Nonlinear terms}$$

Enslaved Equation

Represent all non-critical modes

$$c^{(4)} = \frac{\langle Y,\ \theta(X^{[1:1]}, \overbrace{X^{[0:2]}, X^{[2:2]}, X^{[1:3]}, X^{[3:3]}, X^{[2:4]}, X^{[0:4]}})}{\langle Y, X^{[1:1]} \rangle}$$

Other perturbation methods can be used:

e.g. Amplitude expansion method and multiple scale analysis

1st Landau Coefficient

Shukla & Alam (JFM 2011a)

Linear Problem $$LX^{[1;1]} = c^{(0)} X^{[1;1]}$$

Second Harmonic $$L_{22} X^{[2;2]} = G_{22}$$

Distortion to mean flow $$L_{02} X^{[0;2]} = G_{02}$$

Distortion to fundamental

$$L_{13} X^{[1;3]} = c^{(2)} X^{[1;1]} + G_{13}$$

Analytically solvable

Analytical expression of first Landau coefficient

$$c^{(2)} = \frac{\phi^a G_{13}^1 + u^a G_{13}^2 + v^a G_{13}^3 + T^a G_{13}^4}{\phi^a \phi^{[1;1]} + u^a u^{[1;1]} + v^a v^{[1;1]} + T^a T^{[1;1]}}$$

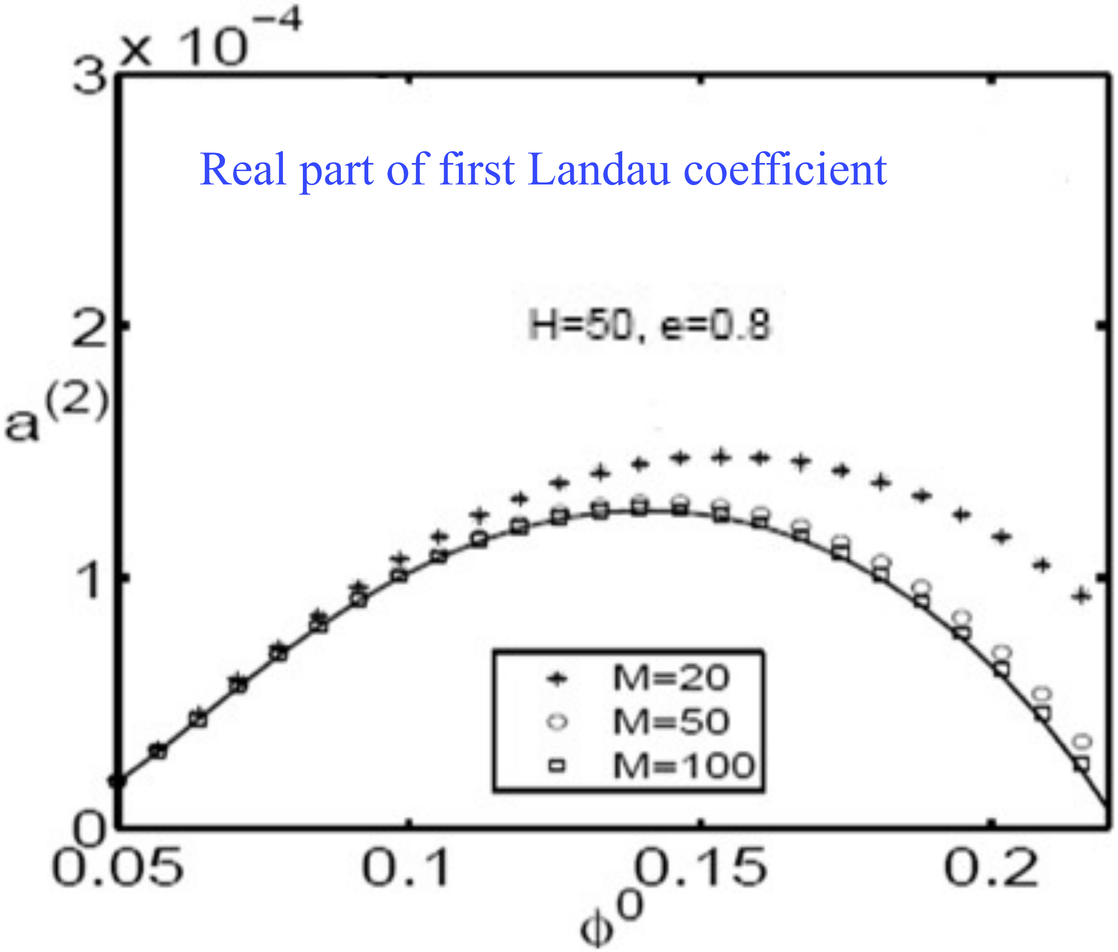
Analytical solution exists.

We have also developed a spectral based numerical code to calculate Landau coefficients.

Numerical Method: comparison with analytical solution

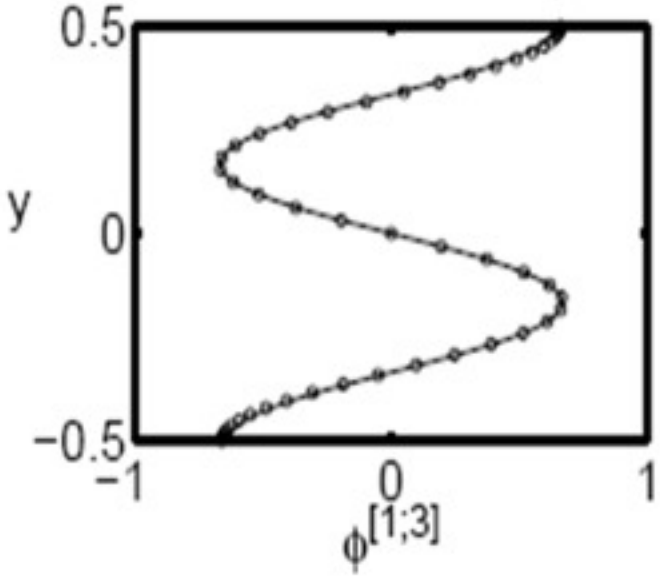
Shukla & Alam JFM (2011a)

Spectral collocation method, SVD for inhomogeneous eqns. & Gauss-Chebyshev quadrature for integrals.



Real part of first Landau coefficient

Distorted density eigenfunction



This validates spectral-based numerical code.

Equilibrium Amplitude and Bifurcation

Cubic Landau Eqn

$$\boxed{\frac{dZ}{dt} = c^{(0)}Z + c^{(2)}Z\,|\,Z\,|^2}$$

$$Z = Ae^{i\theta}$$

$$\frac{dA}{dt} = a^{(0)}A + a^{(2)}A^3,$$

Real amplitude eqn.

$$\frac{d\theta}{dt} = b^{(0)} + b^{(2)}A^2$$

Phase eqn.

Cubic Solution

$$\frac{dA}{dt} = 0 \quad \Rightarrow \quad \boxed{A = 0, \quad A = \pm\sqrt{-\frac{a^{(0)}}{a^{(2)}}}}$$

Supercritical Bifurcation $a^{(0)} > 0, a^{(2)} < 0$

Subcritical Bifurcation $a^{(0)} < 0, a^{(2)} > 0$

$\boxed{b^{(0)} = 0}$

Pitchfork (stationary) bifurcation

$\boxed{b^{(0)} \neq 0}$

Hopf (oscillatory) bifurcation

Phase Diagram

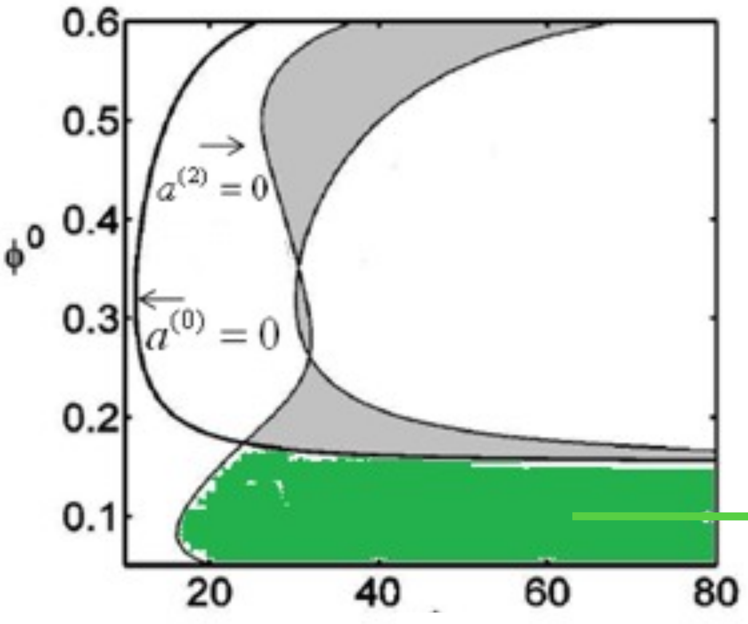
Constitutive equations are function of radial distribution function (RDF)

ϕ^0 : Mean Density

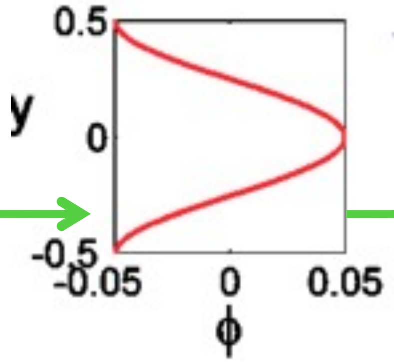
ϕ_m : Maximum Mean Density

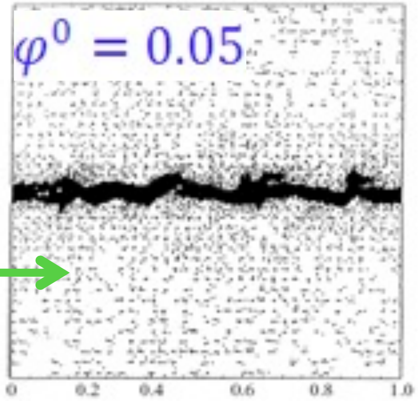
H : Couette Gap

$$\chi(\phi) = \frac{1}{1-(\phi/\phi_m)^{1/3}}$$



$$H^* = H\beta^{-1}\sqrt{(1-e^2)}$$





Shearbanding in dilute flows

⬇

This agrees with MD simulations of
Tan & Goldhirsch 1997

Nonlinear Stability theory and MD simulations both support gradient banding in 2D-GPCF ***(PRL 2009)***

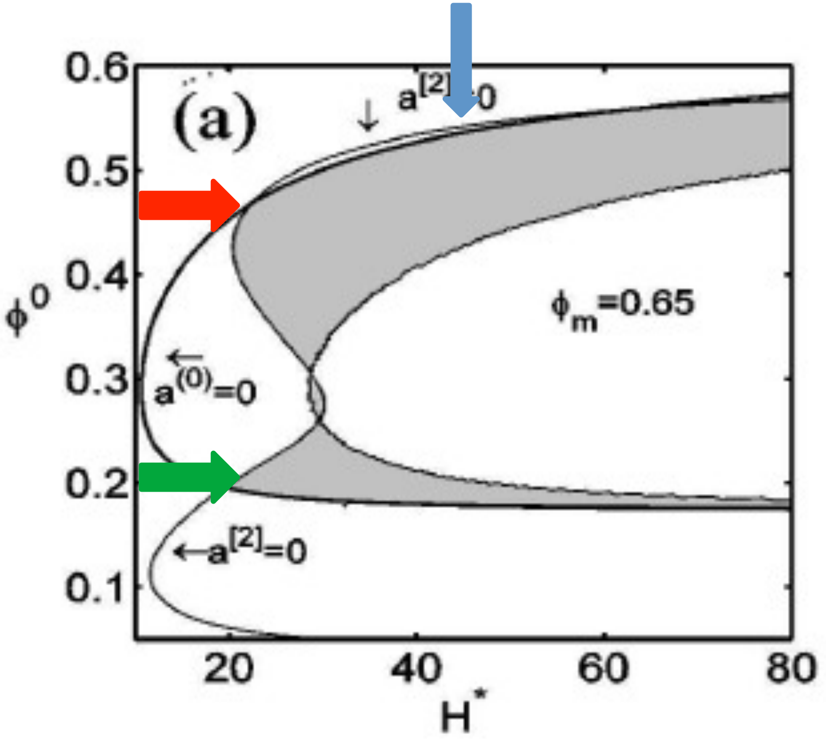
Cont…

(JFM 2011a)

Carnahan-Starling RDF

$$\chi(\phi) = \frac{1-\phi/2}{(1-\phi/\phi_m)^3}$$

Change of constitutive relations lead to three degenerate points



——— Stable Solutions

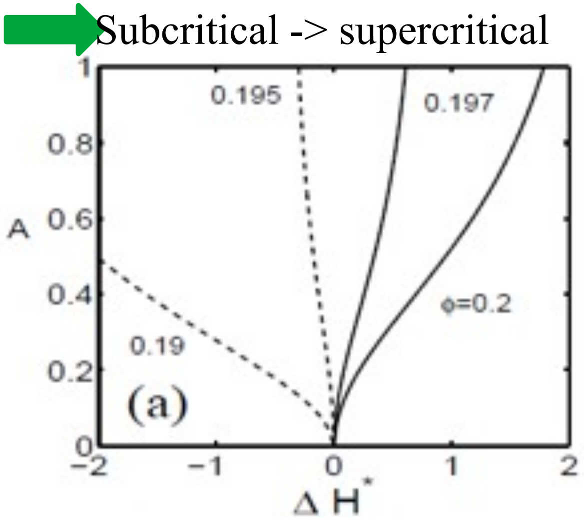
- - - - Unstable Solutions

0.559

Subcritical -> supercritical

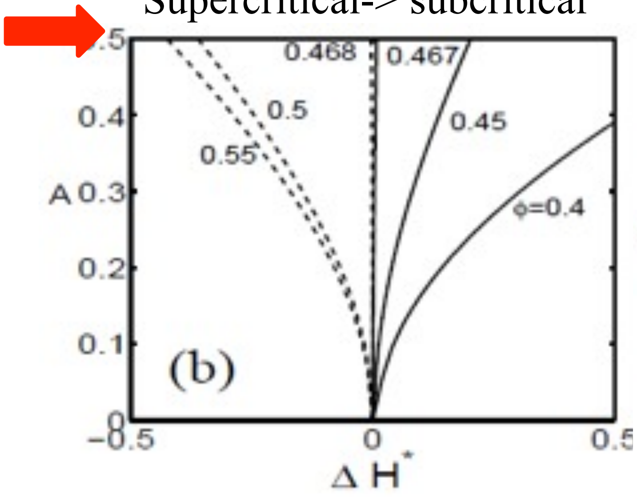
0.196

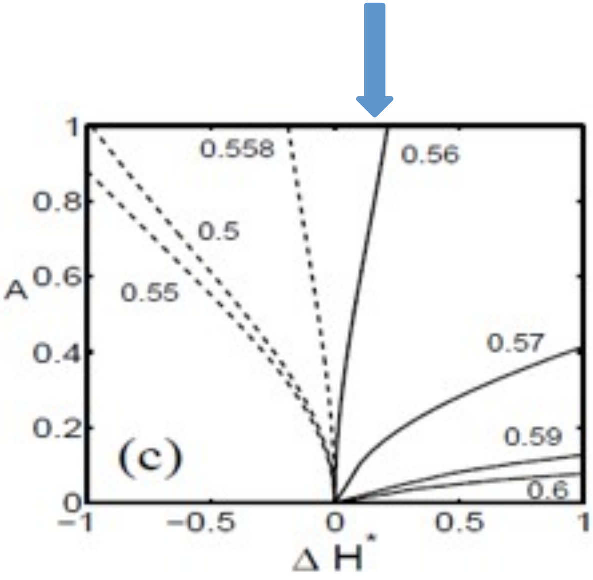
Subcritical -> supercritical



0.467

Supercritical-> subcritical





Paradigm of Pitchfork Bifurcations

Supercritical

$$\phi^{0} > \phi_c^{s2}$$

Subcritical

$$\phi_c^{s1} < \phi^{0} < \phi_c^{s2} \approx 0.559$$

Supercritical

$$\phi_c^{s} < \phi^{0} < \phi_c^{s1} \approx 0.467$$

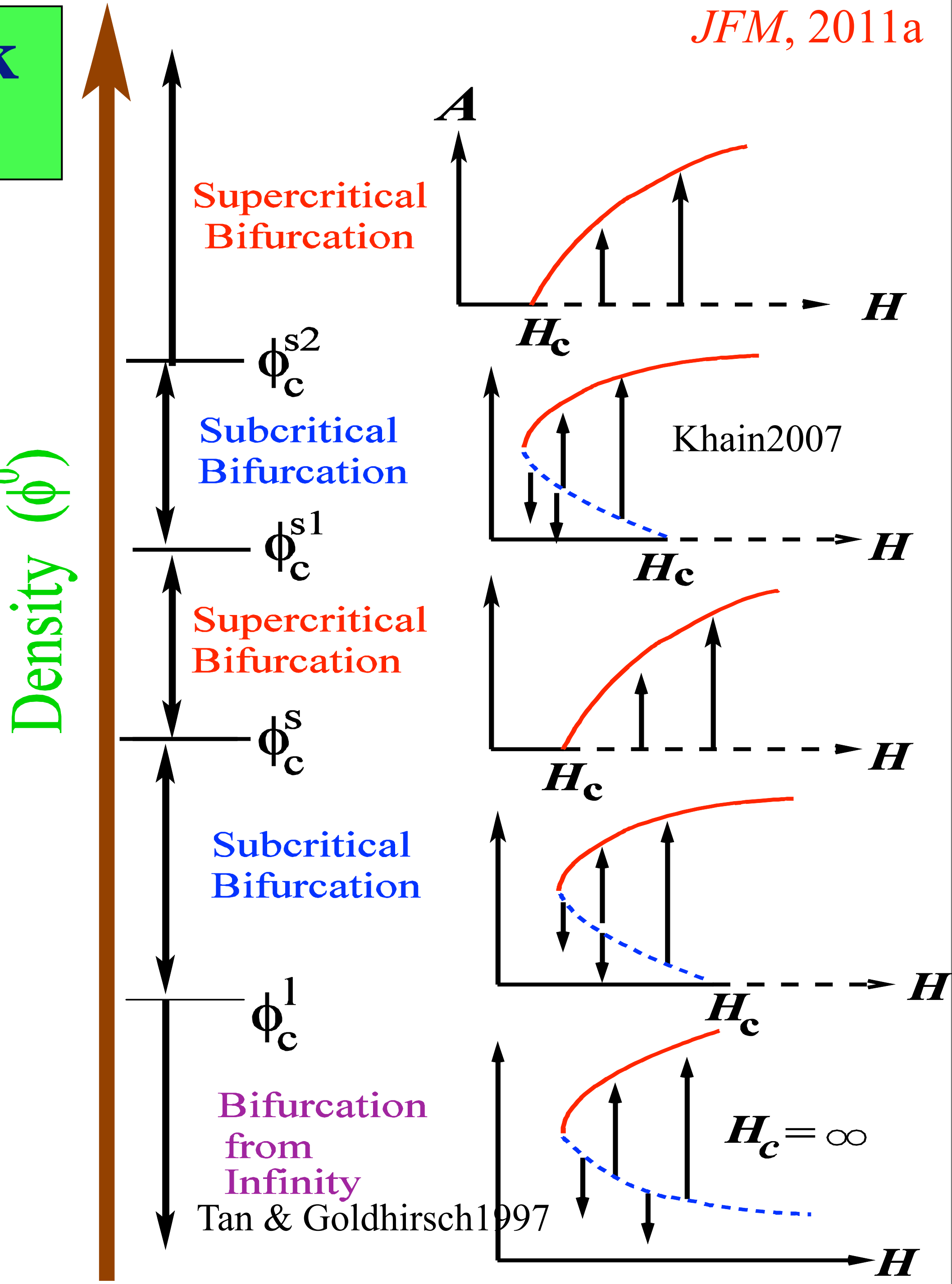
Subcritical

$$\phi_c^{l} < \phi^{0} < \phi_c^{s} \approx 0.196$$

Bifurcation from infinity

$$\phi^{0} < \phi_c^{l} \approx 0.174$$

JFM, 2011a



Conclusions

➢Problem is analytically solvable.

- **Order-parameter equation i.e. Landau equation** describes shear-banding transition in a sheared granular fluid.

- Landau coefficients suggest that there is a "sub-critical" (bifurcation from infinity) finite amplitude instability for "dilute" flows even though the dilute flow is stable according to linear theory.

- This result agrees with previous MD-simulation of gPCF.

- gPCF serves as a **paradigm** of pitchfork bifurcations.

- Analytical solutions have been obtained.

- An spectral based numerical code has been validated.

References: Shukla & Alam (2011a), J. Fluid Mech., vol ***666****, 204-253*
Shukla & Alam (2009) Phys. Rev. Lett., vol ***103*** *, 068001.*

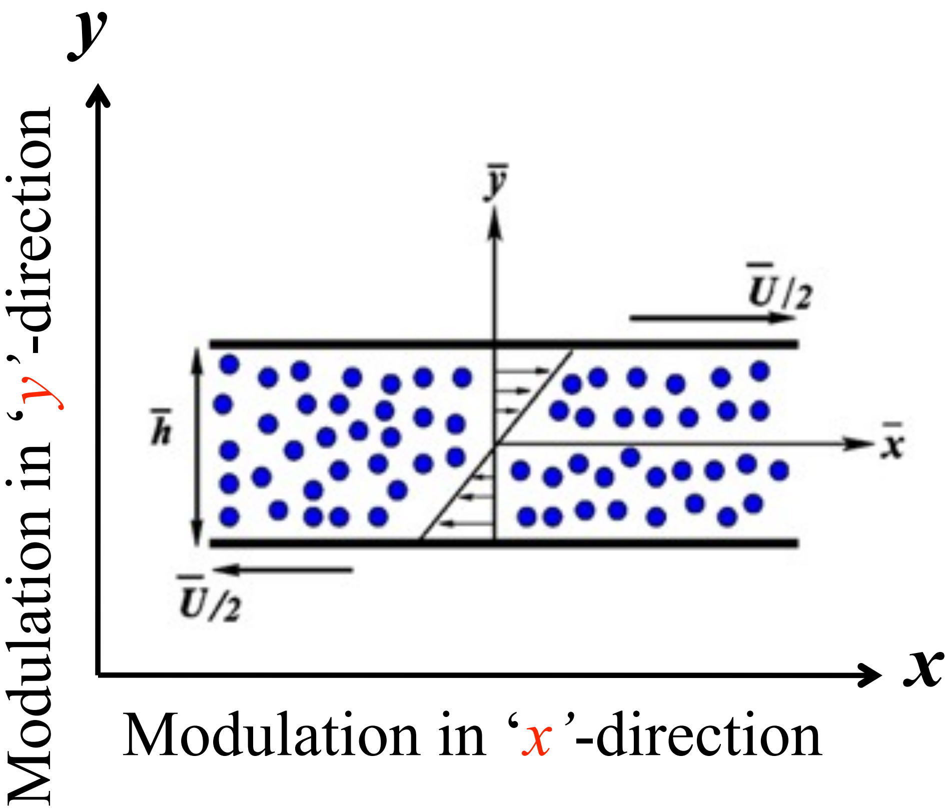
``Gradient-banding” and Saturn’s Ring?

- **Self gravity, corriolis and tidal forces?...**

References: Schmitt & Tscharnuter (1995, 1999) Icarus
Salo, Schmidt & Spahn (2001) Icarus,
Schmidt & Salo (2003) Phys. Rev. Lett.

Patterns in 2D-gPCF

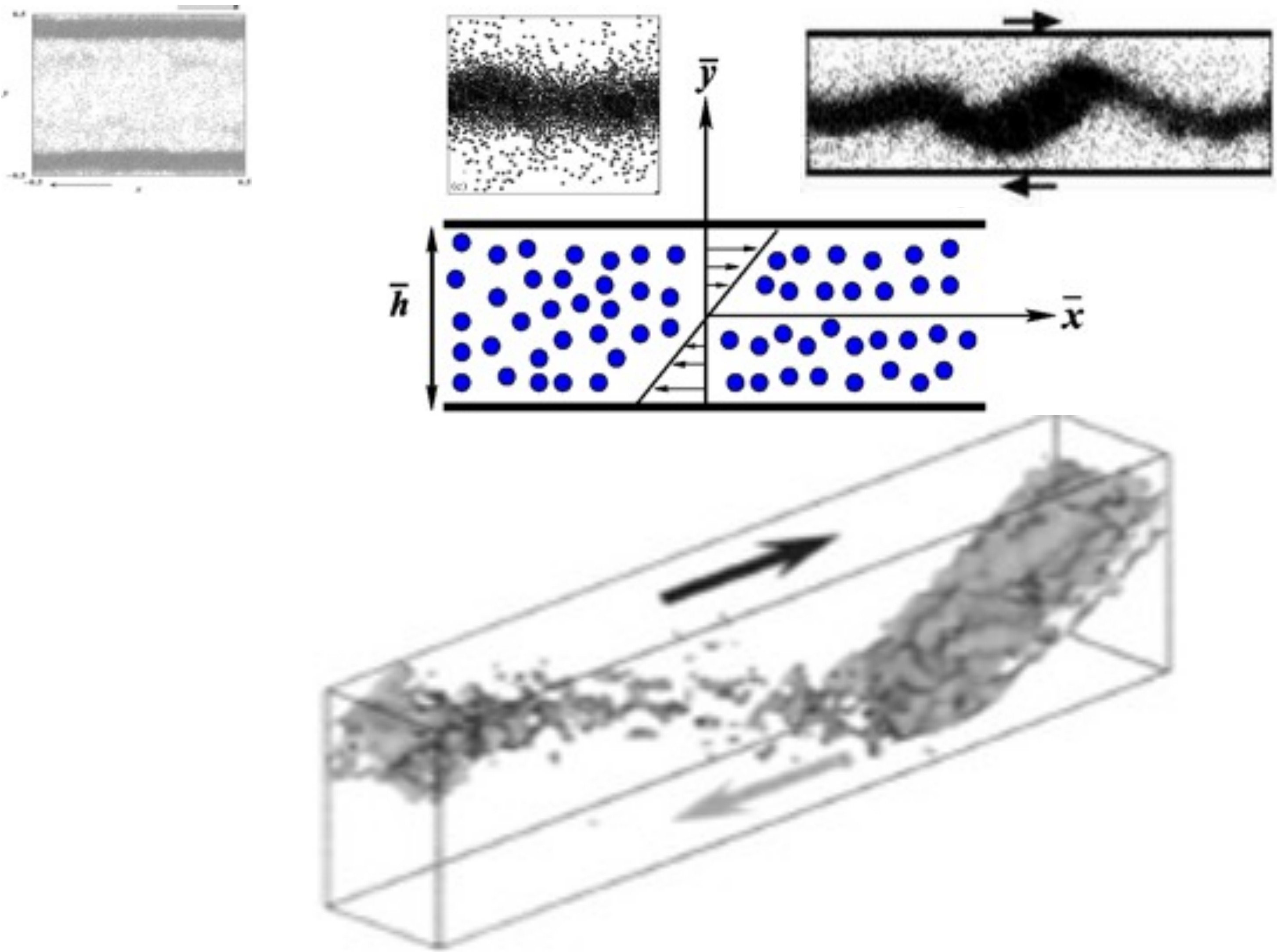
Shukla & Alam , JFM (2011b) vol. 672, 147-195



Flow is unstable due to stationary and traveling waves, leading to particle clustering along the flow and gradient directions (*Alam 2006*)

Particle Simulations of Granular PCF

(Conway and Glasser 2006)



$$\frac{dA}{dt} = f(A,t)$$

$$\frac{\partial A}{\partial t} + a_1 \frac{\partial A}{\partial x} + a_2 \nabla^2 A = g(A,t,x,\cdots)$$

Amplitude Expansion Method

(Stuart, Watson 1960, Reynolds and Potter 1967, Shukla & Alam, JFM 2011a)

A : Real amplitude

$k \leq n$

$n \geq 1$

$$X' = \sum X^{(\mathrm{k})}(y\,,t)\,\mathrm{e}^{i\,k\,(\,k_x\,x\,+\,\grave{u}\,t)} + c.c.$$

$$X^{(\mathrm{k})}(y,t) = A^{(n)} X^{[k;n]}(y)$$

Assumption

$$A^{-1}\frac{dA}{dt} = a^{(0)} + Aa^{(1)} + A^2 a^{(2)} + \mathrm{L}$$

$$\grave{u} + \frac{d\grave{u}}{dA}\left(t\frac{dA}{dt}\right) = b^{(0)} + Ab^{(1)} + A^2 b^{(2)} + \mathrm{L}$$

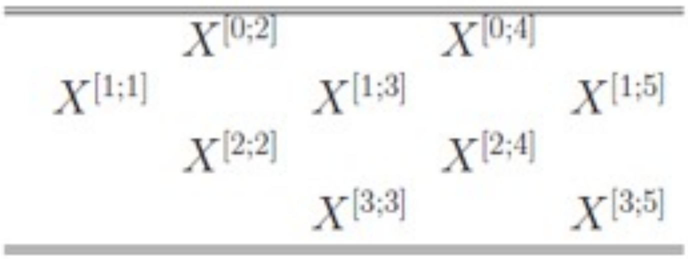
$$L_{kn} X^{[k;n]} = -c^{[\mathrm{n-1}]} X^{[1;1]} \ddot{a}_{k1} + G_{kn}$$

$$L_{kn} = (na^{(0)} + ikb^{(0)})I - L_k$$

$$c^{[\mathrm{n-1}]} = a^{[\mathrm{n-1}]} + ib^{[\mathrm{n-1}]}$$

$$G_{kn} = -\left(ma^{[\mathrm{n-m}]} + ikb^{[\mathrm{n-m}]}\right)X^{\{\mathrm{k;n}\}} + E_{kn}/(1 + \ddot{\mathrm{a}}_{\mathrm{k0}}) + F_{kn}$$

$c^{[n-1]}$ Landau coefficient



$$c^{[\mathrm{n-1}]} = \frac{\int_{-1/2}^{1/2} G_{1n} Y dy}{\int_{-1/2}^{1/2} X^{[1;1]} Y dy}$$

Solvability Condition

For $a^{(0)} \approx 0$

Equivalent to "center manifold reduction"

Linear Theory

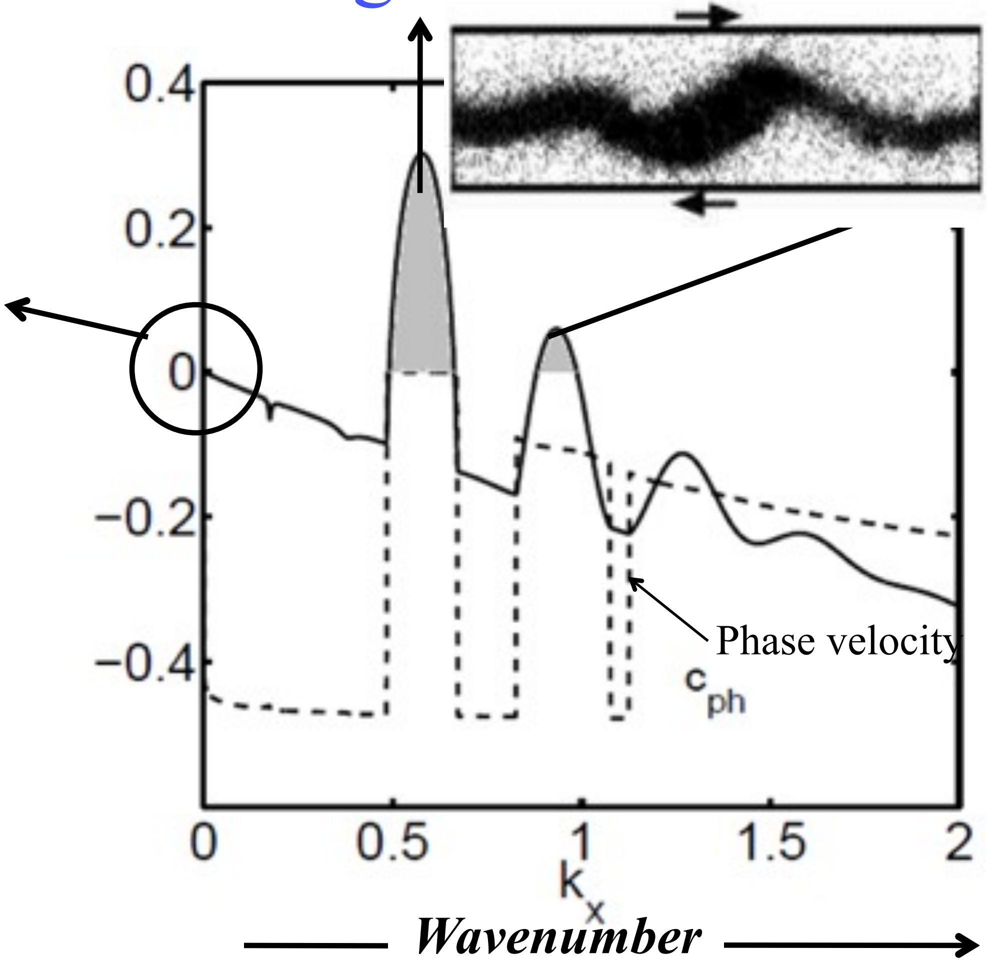
$\phi = 0.2, H = 100, e = 0.8$

1st peak $\boxed{k_x \approx O(1)}$

Standing wave instability

Long-wave instability

$\boxed{k_x \approx 0}$



$\boxed{k_x \approx O(1)}$

2nd peak

Traveling wave instability

$a^{(0)}$ Growth rate

c_{ph} Phase velocity

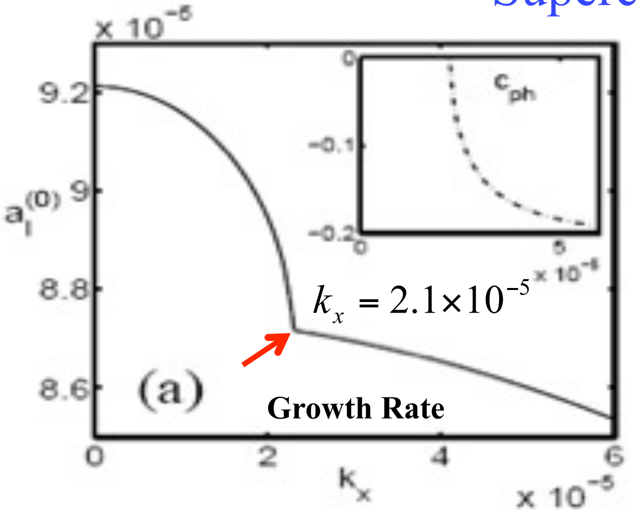
$$\boxed{c_{ph} = -\frac{b^{(0)}}{k_x}}$$

Long-Wave Instabilities

$$\boxed{k_x \approx 0}$$

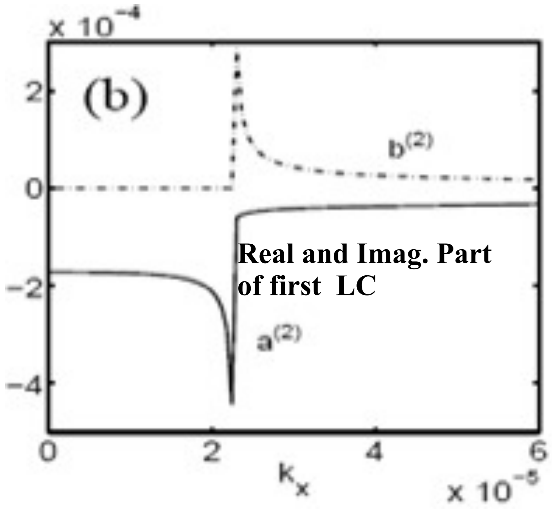
Supercritical pitchfork/Hopf bifurcation

$\phi = 0.2, H = 100, e = 0.8$

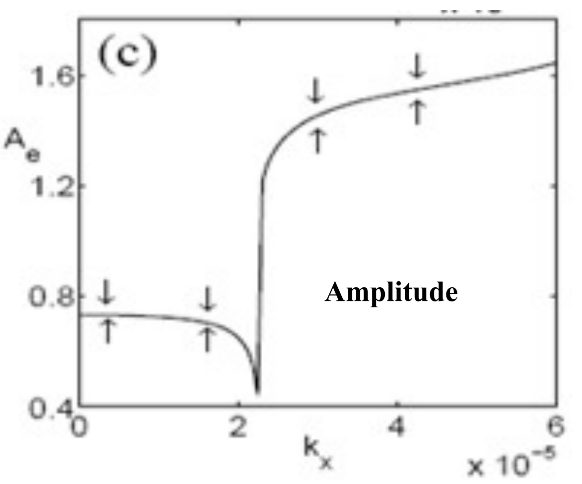


$k_x = 2.1 \times 10^{-5}$

Growth Rate



Real and Imag. Part of first LC

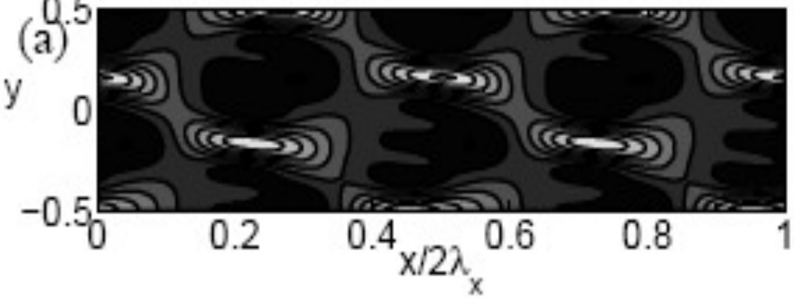


Amplitude

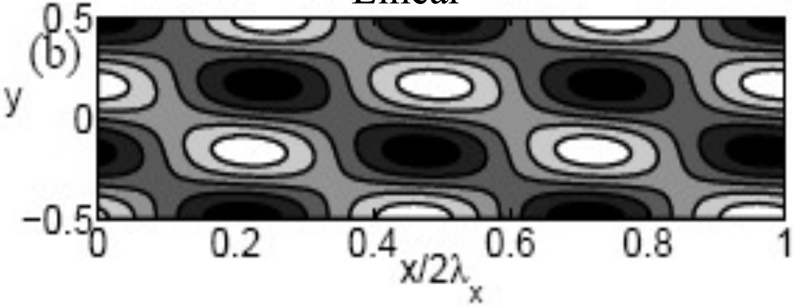
$k_x = 10^{-5}$

SW Density Patterns

Non-linear



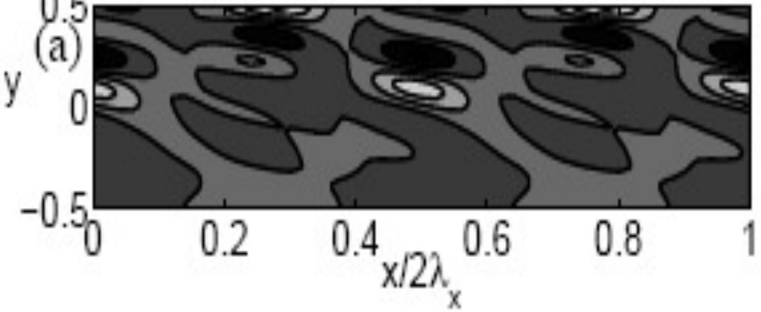
Linear



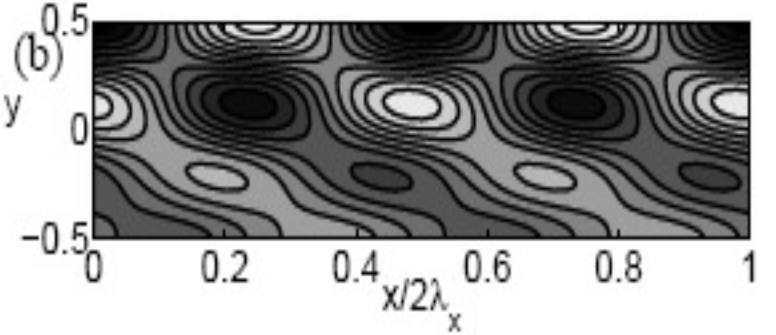
$k_x = 4 \times 10^{-5}$

TW Density Patterns

Non-linear



Linear

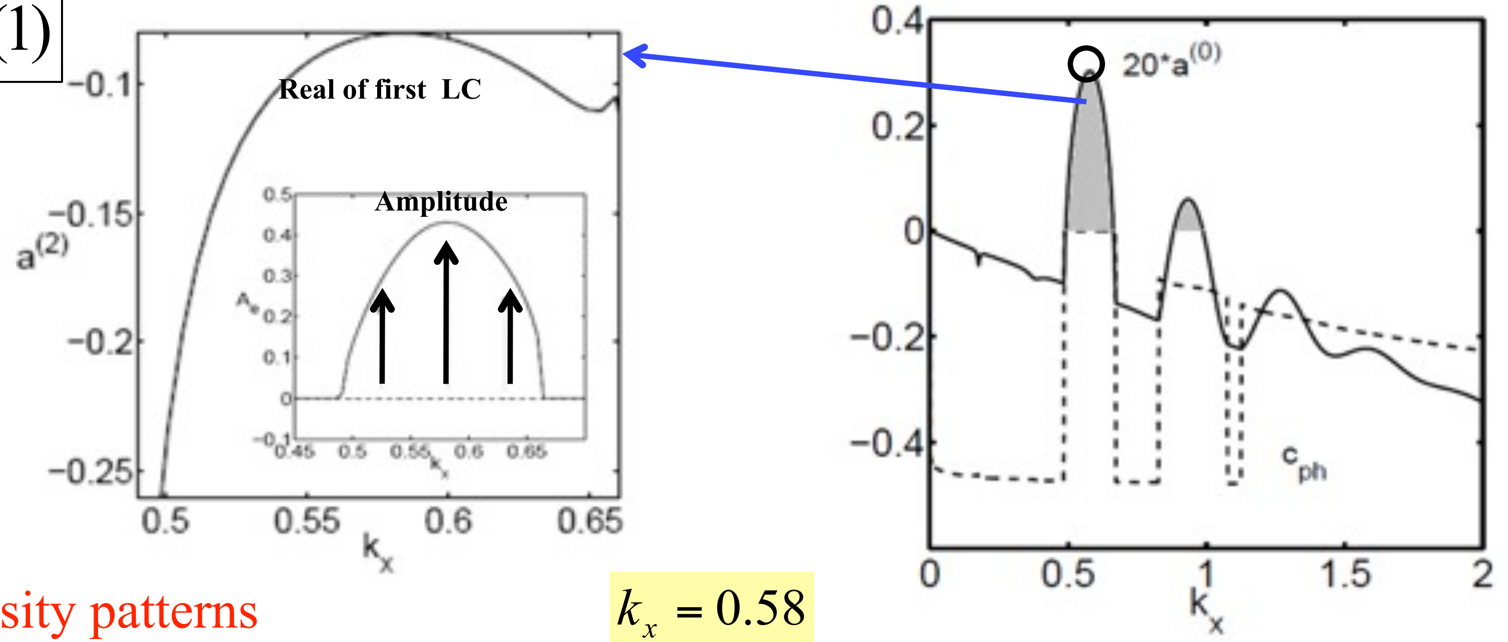


Stationary Instability

$\phi = 0.2, H = 100, e = 0.8$

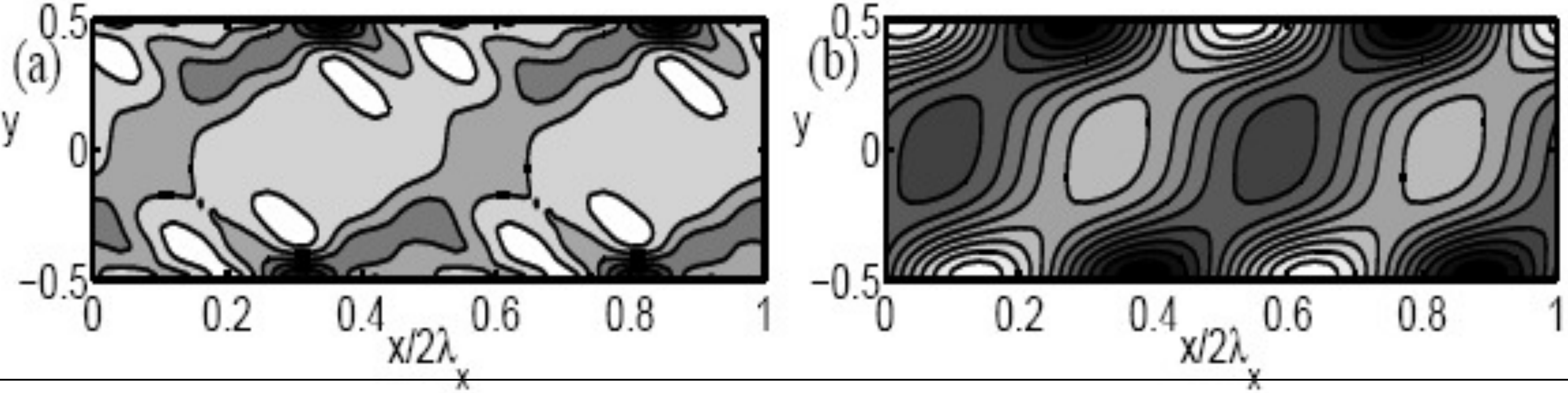
Supercritical pitchfork bifurcation

$k_x \approx O(1)$



SW density patterns

$k_x = 0.58$



Non-linear

Linear

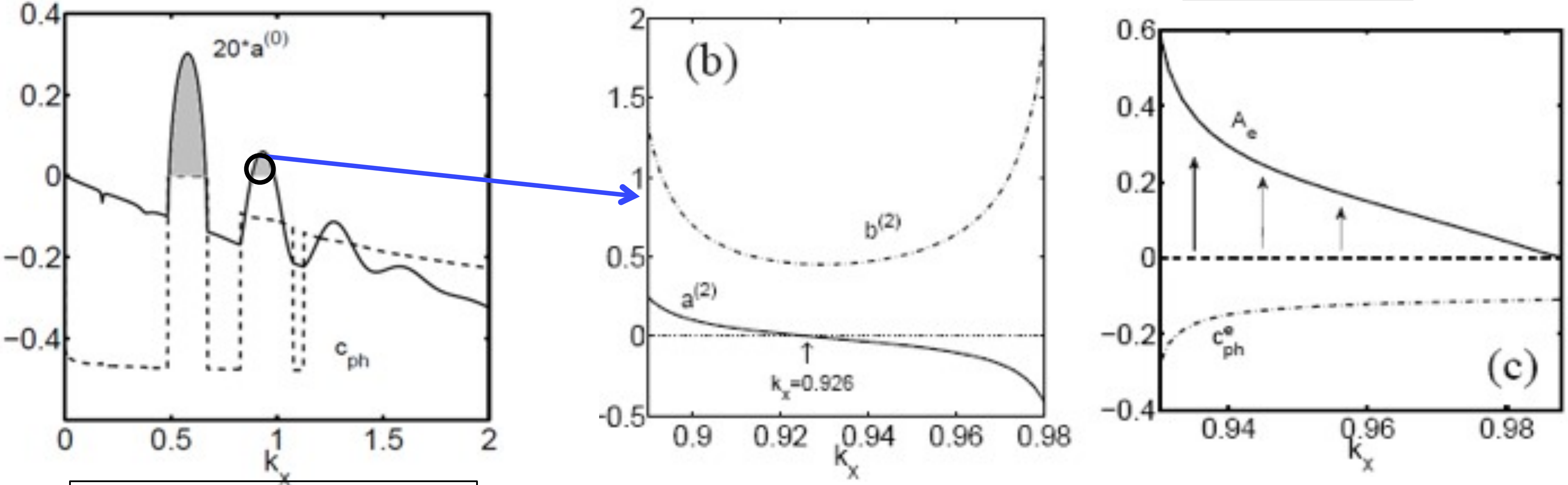
Structural features are different from long-wave stationary instability

Travelling Instabilities

$\phi = 0.2, H = 100, e = 0.8$

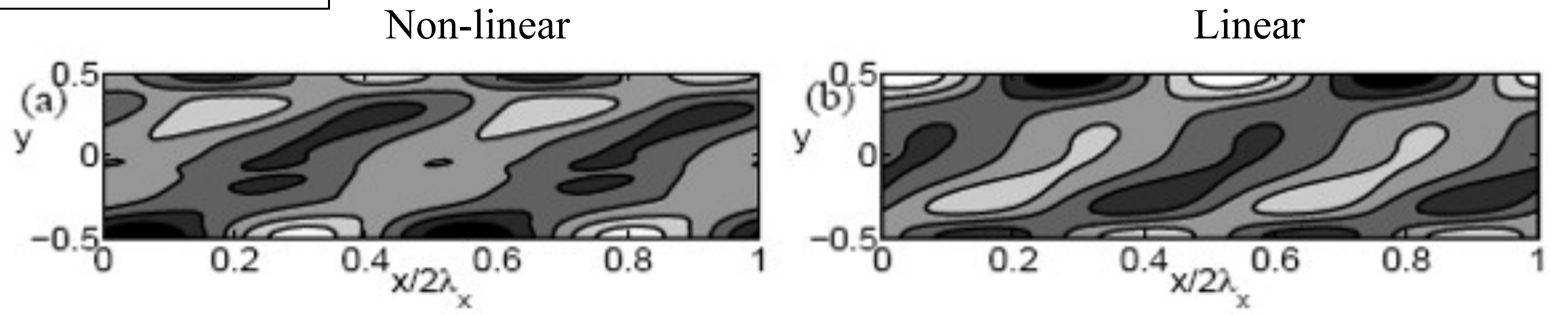
Supercritical Hopf bifurcation

$k_x \approx O(1)$



$$c^e_{ph} = -\frac{\omega}{k_x} = c_{ph} - \frac{b^{(2)}A^2}{k_x}$$

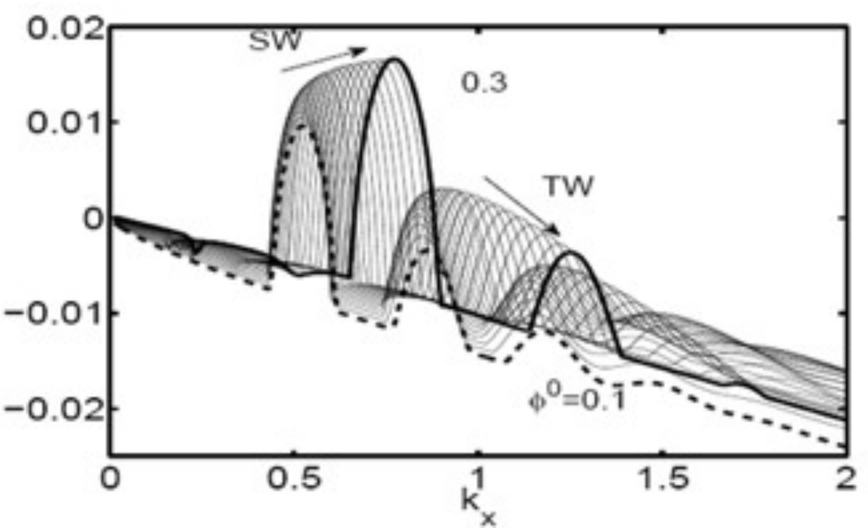
$k_x = 0.935$


Non-linear

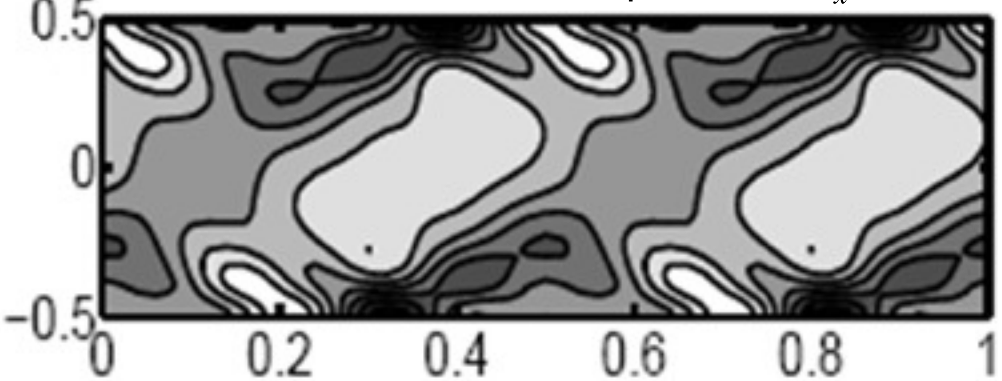
Linear

Nonlinear patterns are slightly affected by nonlinear corrections

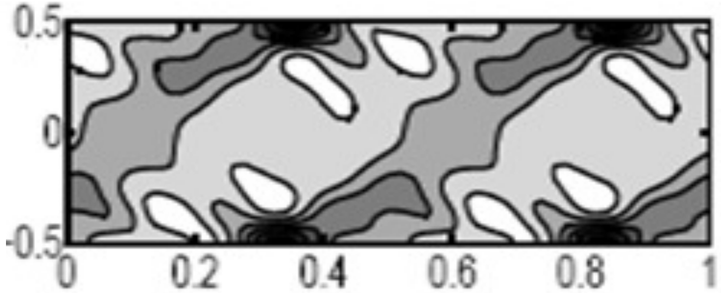
Dominant Stationary Instabilities

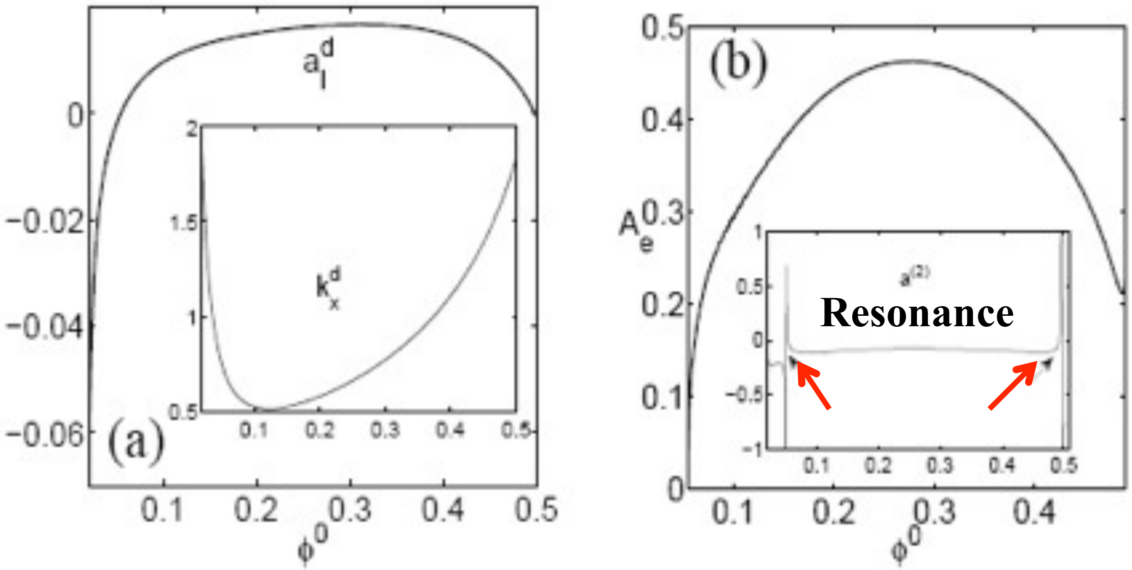


Non-linear $\phi^0 = 0.1, k_x = 0.52$

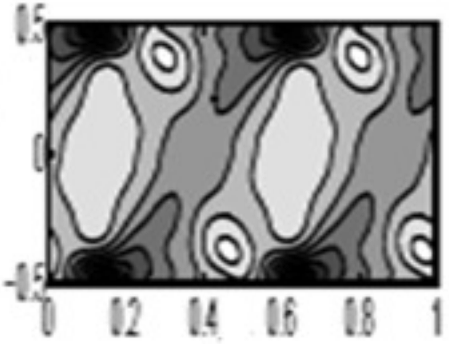


Non-linear $\phi^0 = 0.3, k_x = 0.77$





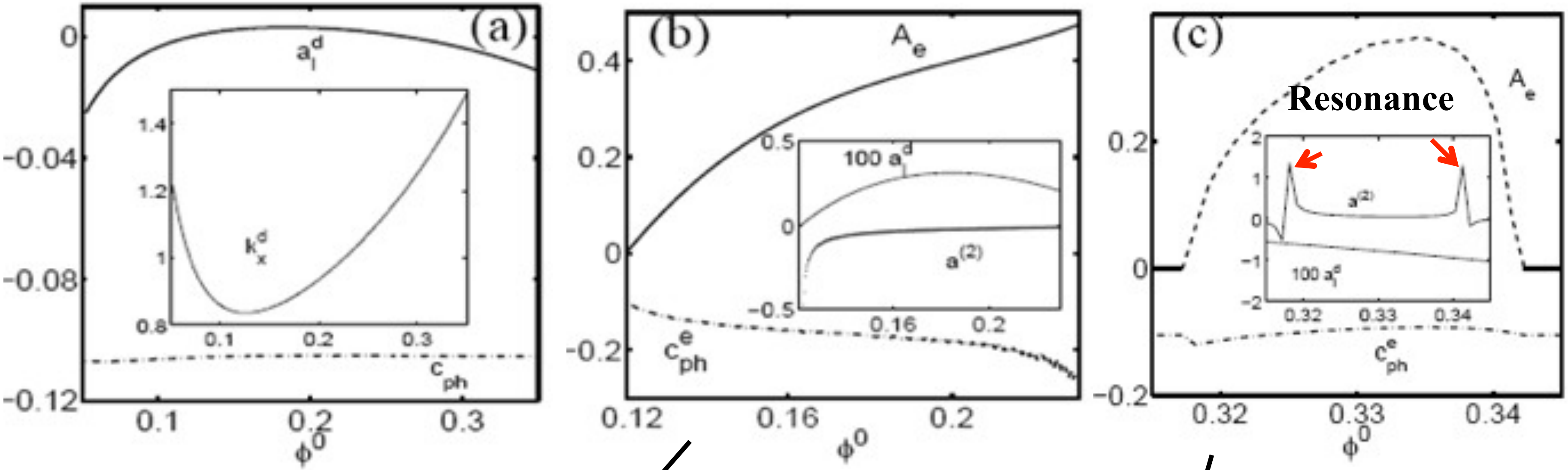
Non-linear $\phi^0 = 0.45, k_x = 1.39$



Density patterns are structurally similar at all densities

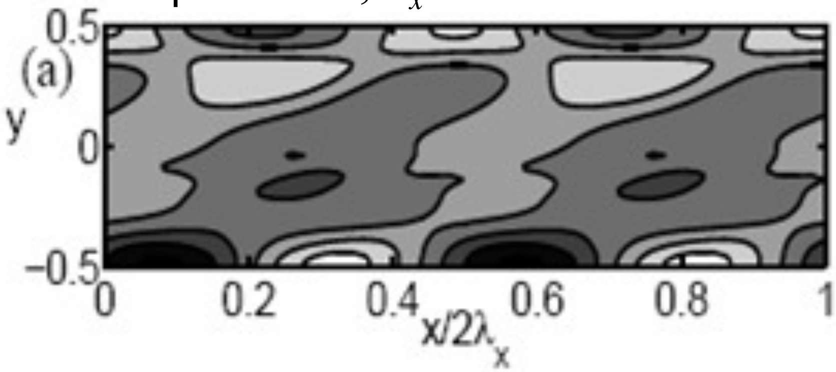
Dominant Traveling Instabilities

Supercritical Hopf Bifurcation Subcritical Hopf Bifurcation



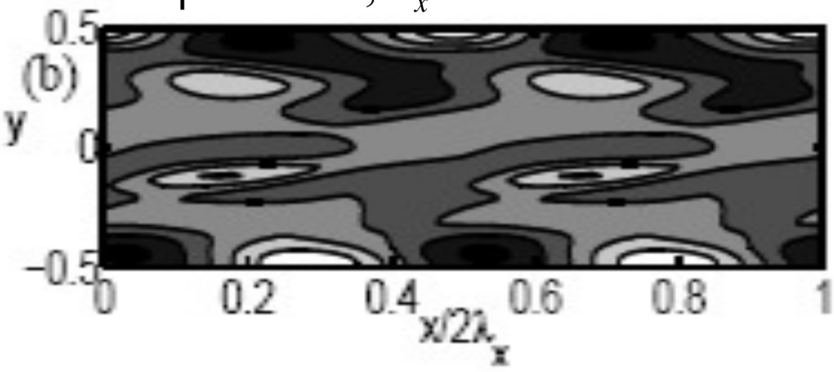
Non-linear

$\phi^0 = 0.15, k_x = 0.85$ **Stable**



Non-linear

$\phi^0 = 0.33, k_x = 0.13$ **Unstable**

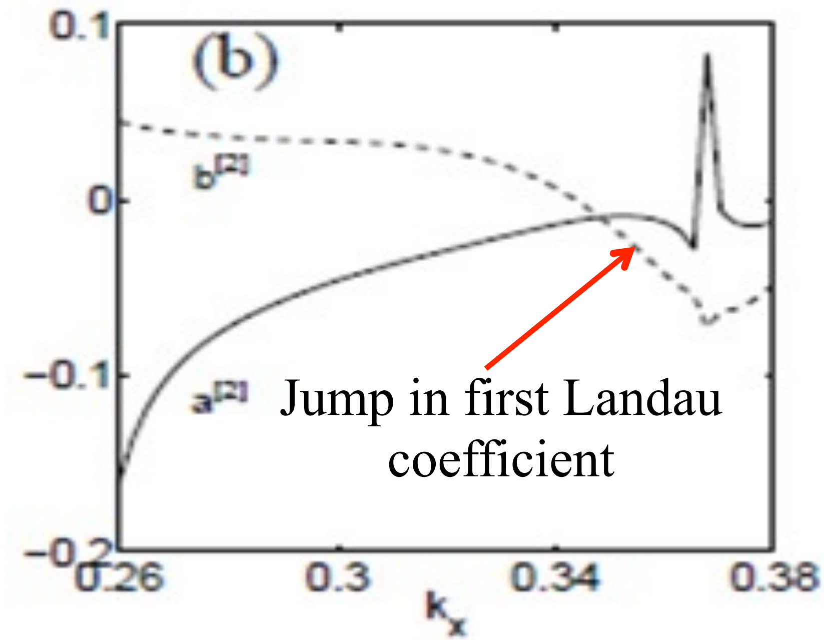


Evidence for Resonance

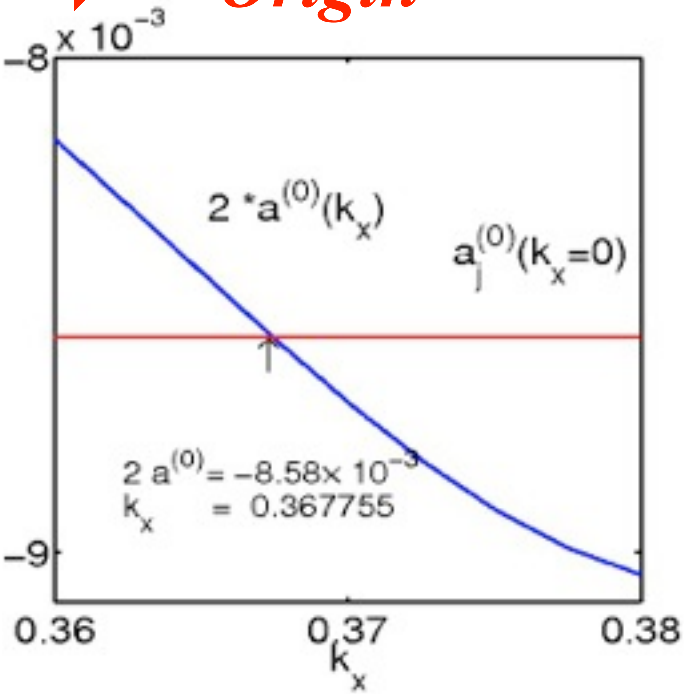
$\phi = 0.2, H = 100, e = 0.8$

Subcritical region

Evidence ← $a^{(0)} < 0$ → *Origin*



Jump in first Landau coefficient



Distortion of mean flow Eqn.

$$\boxed{X^{[0;2]} = L_{02}^{-1} E_{02} = [2a^{(0)} I - L_0]^{-1} E_{02} = \mathit{finite}}$$

Criterion for mean flow resonance

$$2\,a^{(0)}\,(k_x) = a^{(0)}(k_x = 0)$$

Interaction of linear mode with a shear banding mode

Second Harmonic Eqn.

$$\boxed{X^{[2;2]} = L_{22}^{-1} E_{22} = [2c^{(0)} I - L_2]^{-1} E_{22} = \mathit{finite}}$$

Criterion for 1:2 resonance

$$2\,c^{(0)}\,(k_x) = c^{(0)}(2k_x)$$

Evidence for Resonance

Multiple resonance in subcritical region

$$\phi = 0.2,\ H = 100,\ e = 0.8$$

Single mode analysis is not valid at the resonance point

Coupled Landau Equations

$$\frac{dZ_1}{dt} = c_1 Z_1 + \lambda_{11} Z_1 |Z_1|^2 + \lambda_{12} Z_1 Z_2^2 + \lambda_{13} Z_1 Z_2$$

$$\frac{dZ_2}{dt} = c_2 Z_2 + \lambda_{21} Z_2 |Z_1|^2 + \lambda_{22} Z_2^3 + \lambda_{23} |Z_1|^2 + \lambda_{24} Z_2^2$$

Conclusions

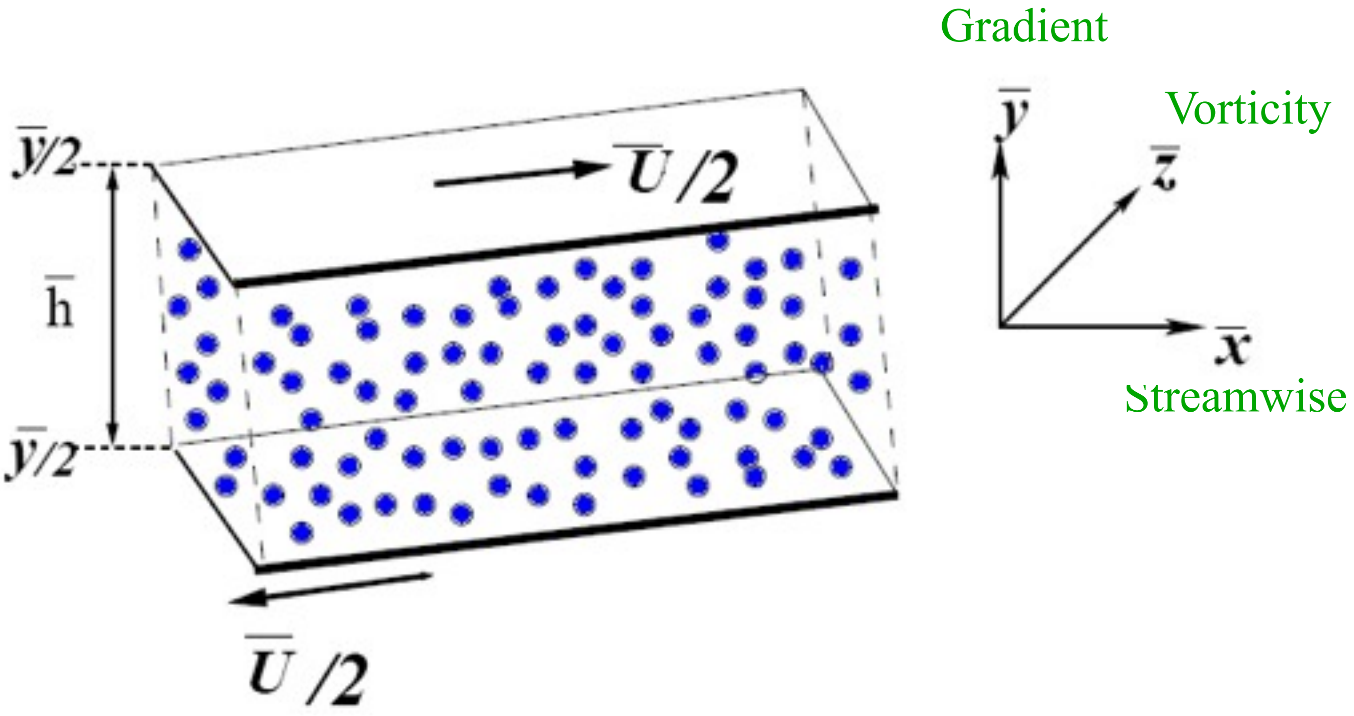
- The origin of nonlinear states at long-wave lengths is tied to the corresponding subcritical / supercritical nonlinear gradient-banding solutions (discussed in 1st Part of talk).
- For the dominant stationary instability nonlinear solutions appear via supercritical bifurcation.
- Structure of patterns of supercritical stationary solutions look similar at any value of density and Couette gap.
- For the dominant traveling instability, there are supercritical and subcritical **Hopf** bifurcations at small and large densities.
- Uncovered mean flow resonance at quadratic order.

References: *Shukla & Alam (2011b), J. Fluid Mech., vol. 672, p. 147-195.*
Shukla & Alam (2011a), J. Fluid Mech., vol ***666****, p. 204-253.*
Shukla & Alam (2009) Phys. Rev. Lett., vol ***103*** *, 068001.*

Vorticity Banding in 3D-gPCF

Pure Spanwise gPCF

$$\frac{\partial}{\partial x} = 0, \quad \frac{\partial}{\partial y} = 0, \quad \frac{\partial}{\partial z} \neq 0$$



Shukla & Alam (2011c) (Submitted)

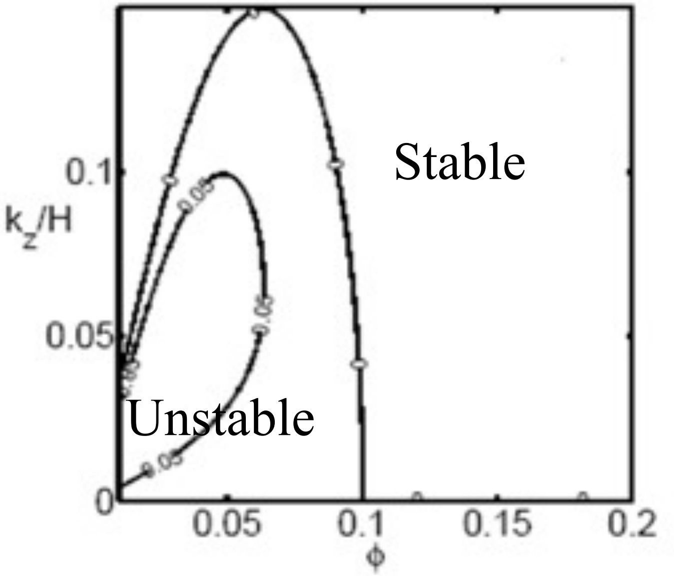
Linear Vorticity Banding

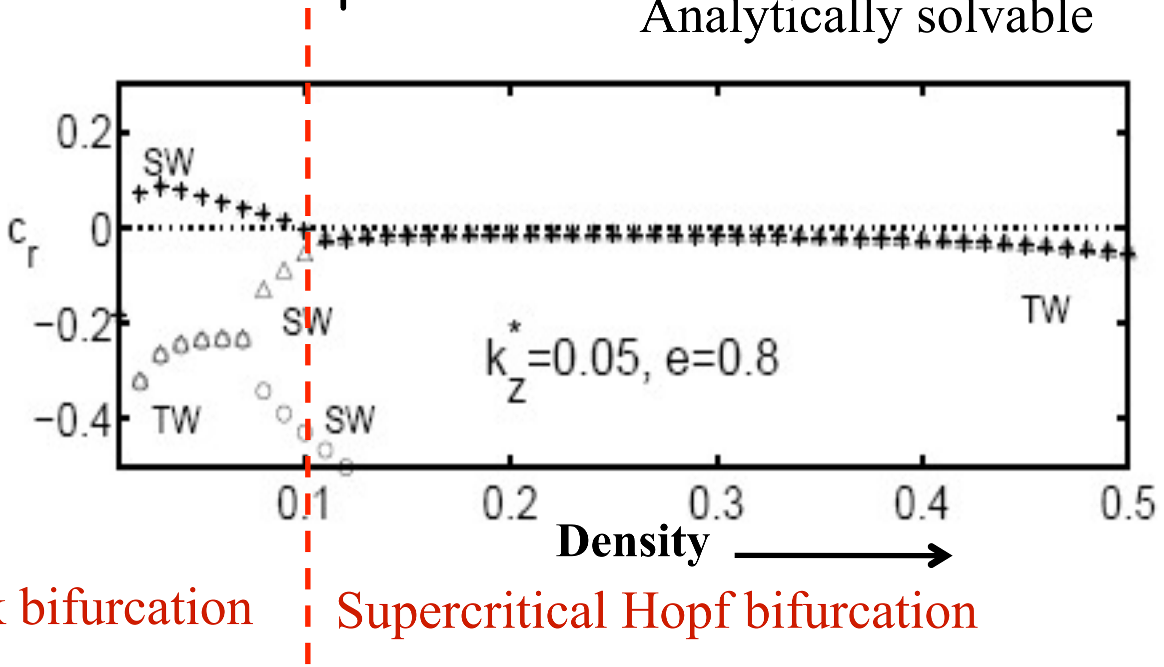
Pure spanwise GPCF

Dispersion relation

$$\left(\omega + \frac{\mu^0 k_z^2}{\phi^0 H^2}\right)^2 \underbrace{(\omega^3 + a_2\omega^2 + a_1\omega + a_0)} = 0$$

Analytically solvable





Pitchfork bifurcation

Supercritical Hopf bifurcation

Gradient-banding modes stationary **modes at all density.**

Vorticity-banding modes stationary **at dilute limit &**
traveling **in moderate-to-dense limit.**

Nonlinear Stability

Shukla & Alam (2011) (Submitted)

Linear Problem $LX^{[1;1]} = c^{(0)} X^{[1;1]}$

Second Harmonic $L_{22} X^{[2;2]} = G_{22}$

Distortion to mean flow $L_{02} X^{[0;2]} = G_{02}$

Distortion to fundamental

$$L_{13} X^{[1;3]} = c^{(2)} X^{[1;1]} + G_{13}$$

Analytically solvable

Analytical expression for first Landau coefficient

$$c^{(2)} = \frac{\phi^a G_{13}^1 + w^a G_{13}^4 + T^a G_{13}^5}{\phi^a \phi^{[1;1]} + w^a w^{[1;1]} + T^a T^{[1;1]}}$$

Adjoint Eigenfunction

(ϕ^a, w^a, T^a)

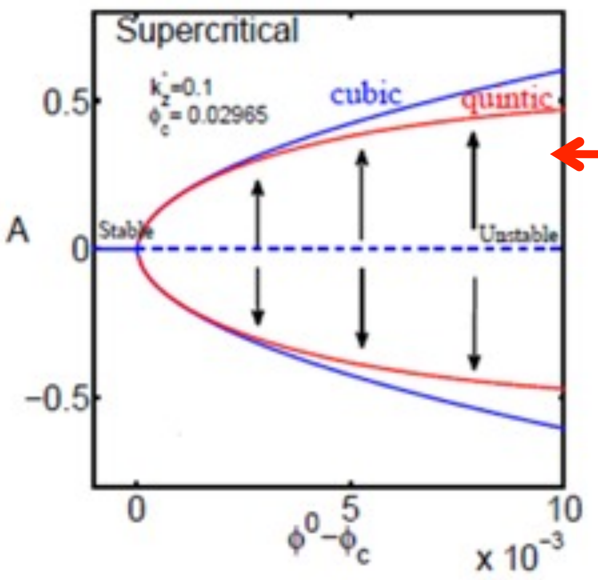
Analytical solution exists at any order in amplitude.

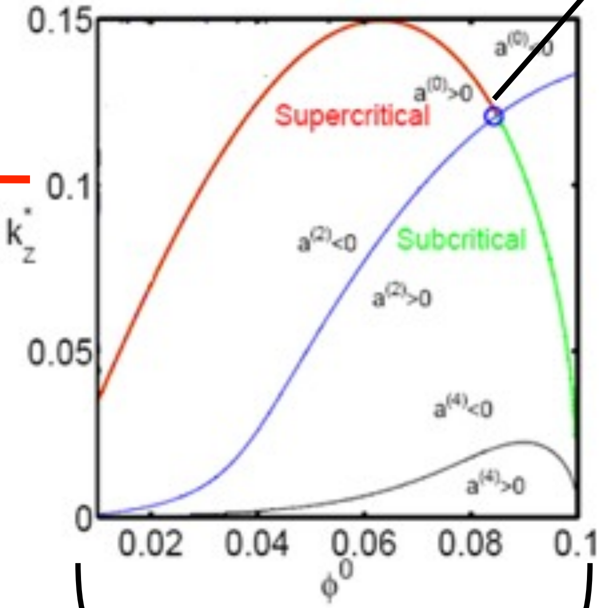
Nonlinear Vorticity Banding

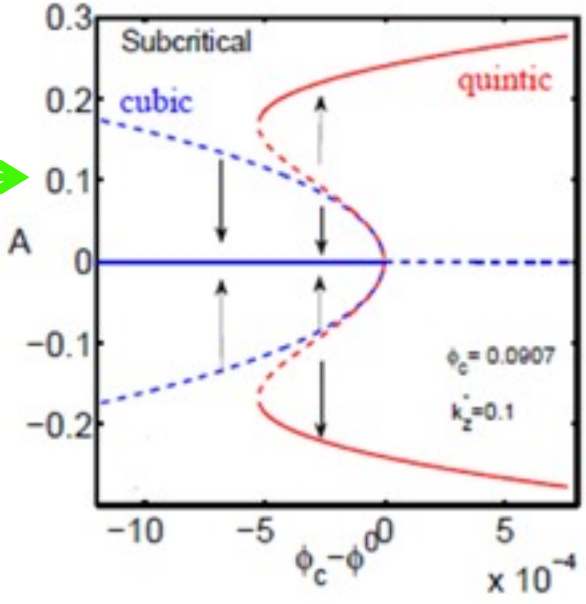
$$A_e = \pm\sqrt{-\frac{a^{(0)}}{a^{(2)}}}$$

$$A_e = \pm\sqrt{\frac{-a^{(2)} \pm \sqrt{(a^{(2)})^2 - 4a^{(0)}a^{(4)}}}{2a^{(4)}}}$$

ϕ^d







$\phi^0 < 0.08$

$\phi^d = 0.08$

$0.08 < \phi^0 \le 0.1$

Supercritical Pithfork Bifurcation

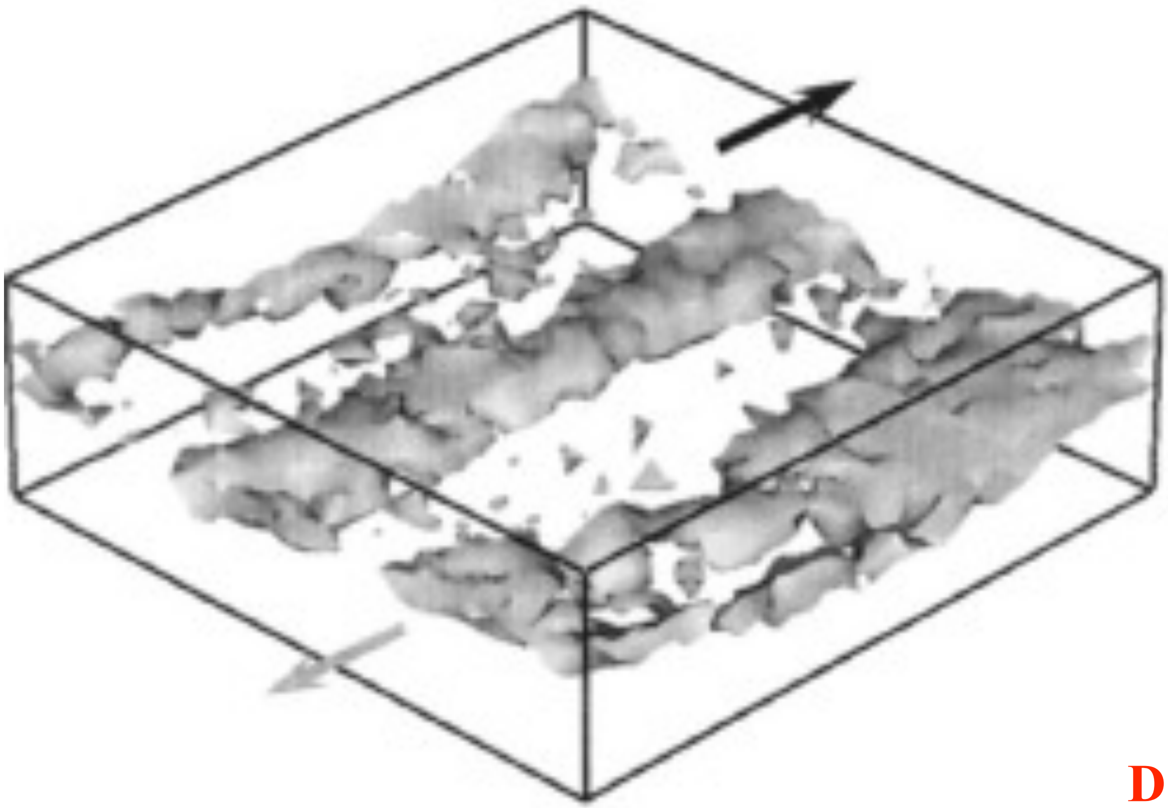
Subcritical Pitchfork Bifurcation

Density

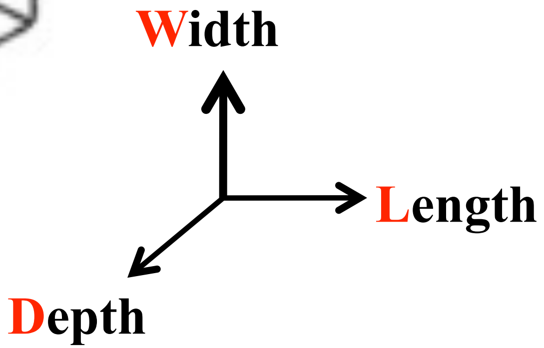
Vorticity Banding in Dilute 3D Granular Flow

(Conway and Glasser. Phys. Fluids, 2004)

Particle density iso-surfaces for $\phi = 0.05, e = 0.6$

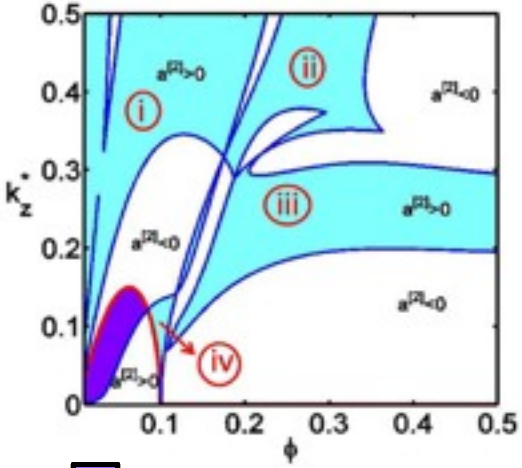


$$\frac{L}{W} = \frac{D}{W} = 3$$



Conclusions

Vorticity Banding



🟪 Supercritical Region

🟦 Subcritical Region

$\boxed{\phi^0 = 0.1}$

Pitchfork Bifurcation

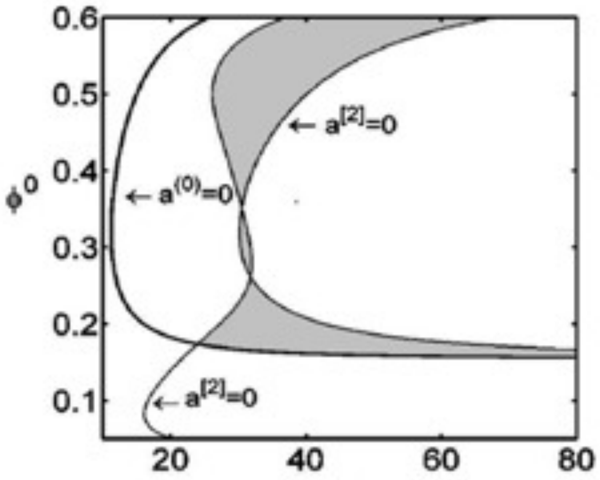
Subcritical and supercritical

Hopf Bifurcation

Subcritical

Density

Gradient Banding



$$H^* = H\beta^{-1}\sqrt{(1-e^2)}$$

Pitchfork Bifurcation

Density

Analytical solution exists at any order.

Higher order nonlinear terms are important to get correct bifurcation scenario.

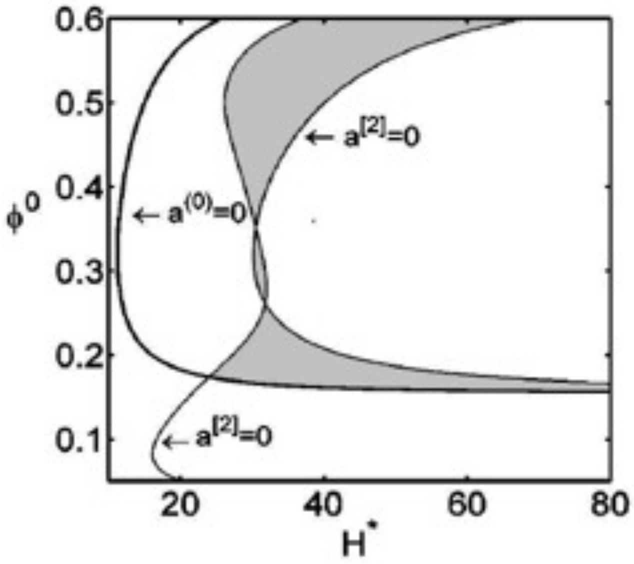
Shukla & Alam (2011 Submitted)

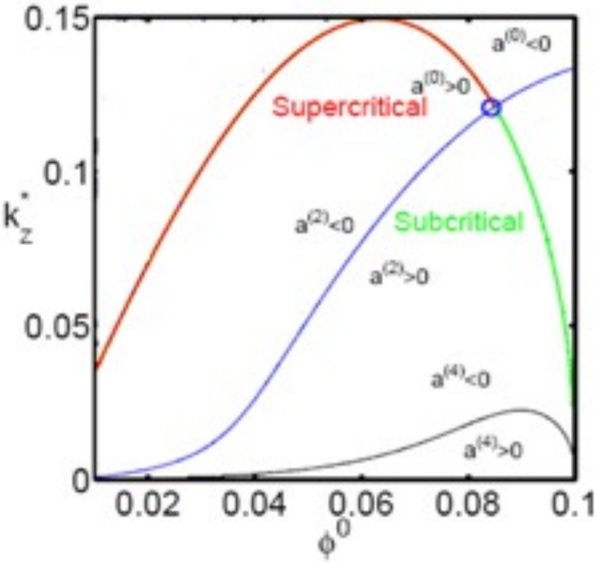
Theory for Mode Interaction

(via Coupled Landau Equations)

Case1

Coupled Landau Equations for non-resonating modes





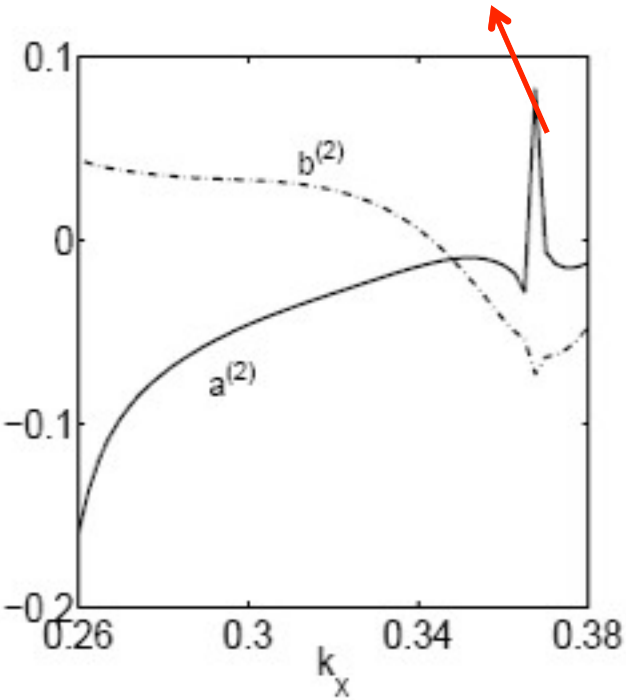
Gradient Banding ⟷ Vorticity Banding

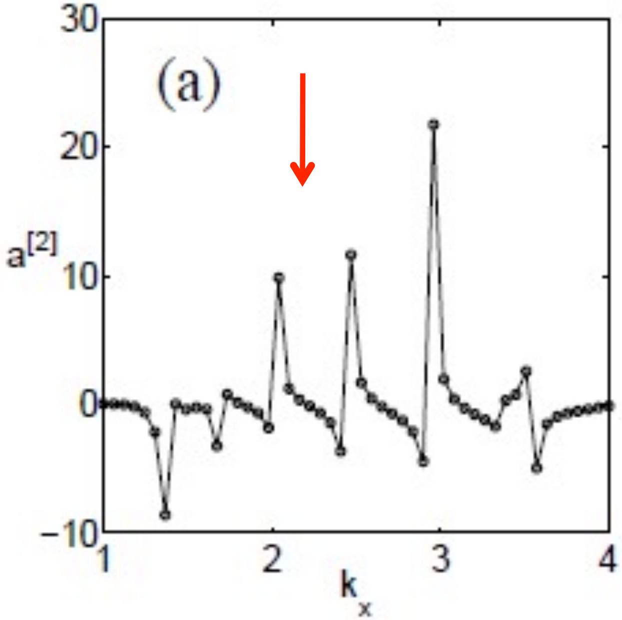
In dilute-regime both gradient and vorticity banding modes exist

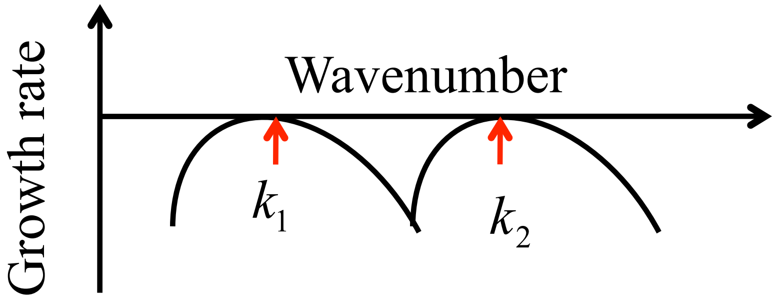
Coupled Landau Equations for resonating modes

Case 2

Single mode analysis fails





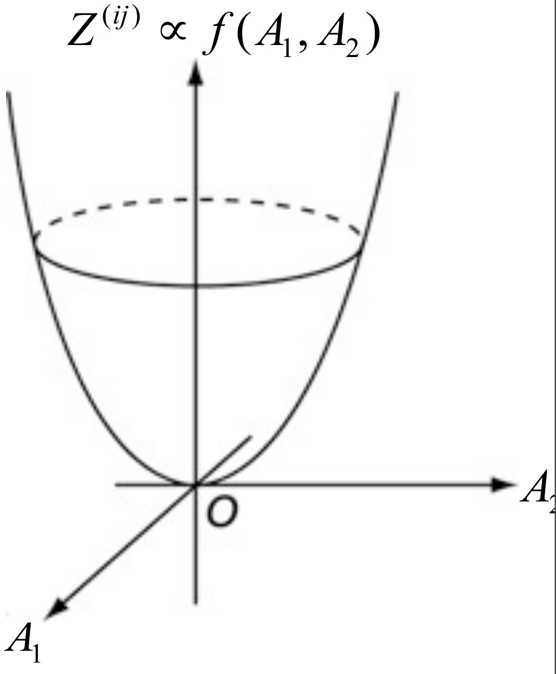


Condition for 1:n resonance.

$$\frac{k_1}{k_2} = \frac{1}{n} \qquad \frac{c_1}{c_2} = \frac{1}{n}$$

Theory for Mode Interaction

Two dimensional Center Manifold



Center-manifold reduction (Carr 1981)

$$\left(\frac{\partial}{\partial t} - L\right) X'(x, y, t) = \sum_{i=2}^{\infty} N_i$$

$$\boxed{L_{01} Y^{[1;1]} = c_2 Y^{[1;1]}}$$

EVP

$$X' = \phi + \psi$$

Amplitude of 2nd mode

$$\phi = \boxed{A_1(t) E_1 X^{[1;1]}} + \boxed{A_2(t) E_2 Y^{[1;1]}} + c.c.$$

Amplitude of 1st mode

EVP

$$\boxed{L_{10} X^{[1;1]} = c_1 X^{[1;1]}}$$

$$E_1 = e^{ik_1 x} \qquad E_2 = e^{ik_2 x}$$

$$\begin{aligned} \lambda_{11} &= \langle \tilde{X}^\dagger, G_{13}^{11} \rangle / \langle \tilde{X}^\dagger, X^{[1;1]} \rangle, \\ \lambda_{21} &= \langle \tilde{Y}^\dagger, G_{13}^{21} \rangle / \langle \tilde{Y}^\dagger, Y^{[1;1]} \rangle, \end{aligned}$$

$$\lambda_{12} = \langle \tilde{X}^\dagger, G_{13}^{12} \rangle / \langle \tilde{X}^\dagger, X^{[1;1]} \rangle$$

$$\lambda_{22} = \langle \tilde{Y}^\dagger, G_{13}^{22} \rangle / \langle \tilde{Y}^\dagger, Y^{[1;1]} \rangle.$$

$$X' = \sum_{k=-\infty}^{\infty} (X^{(k)} E_1^k + Y^{(k)} E_2^k) + \sum_{i,j \geq 0, i=j \neq 0}^{\infty} Z^{(ij)} E_1^i E_2^j + c.c.$$

$$\left(\frac{d}{dt} - c_1\right) A_1 X^{[1;1]} = G_{13}^{11} A_1 |A_1|^2 + G_{13}^{12} A_1 |A_2|^2 + L$$

$$\left(\frac{d}{dt} - c_2\right) A_2 Y^{[1;1]} = G_{13}^{21} A_2 |A_1|^2 + G_{13}^{22} A_2 |A_2|^2 + L$$

$$\frac{dA_1}{dt} = c_1 A_1 + \lambda_{11} A_1 |A_1|^2 + \boxed{\lambda_{12} A_1 |A_2|^2}$$

$$\frac{dA_2}{dt} = c_2 A_2 + \boxed{\lambda_{21} A_2 |A_1|^2} + \lambda_{22} A_2 |A_2|^2$$

Mode Interaction and Coupled Landau Eqn.

Coupled Landau Equation
Non-resonating modes

$$\begin{aligned} \frac{dA_1}{dt} &= c_1 A_1 + \lambda_{11} A_1 |A_1|^2 + \boxed{\lambda_{12} A_1 |A_2|^2} \\ \frac{dA_2}{dt} &= c_2 A_2 + \boxed{\lambda_{21} A_2 |A_1|^2} + \lambda_{22} A_2 |A_2|^2 \end{aligned}$$

Coupled Landau Equation
1:2 resonance

$$\frac{k_{x_1}}{k_{x_2}} = \frac{1}{2}, \quad \frac{c_1}{c_2} = \frac{1}{2}$$

$$\begin{aligned} \frac{dA_1}{dt} &= c_1 A_1 + \lambda_{11} A_1 |A_1|^2 + \boxed{\lambda_{12} A_1 |A_2|^2} + \boxed{\lambda_{13} \tilde{A}_1 A_2} \\ \frac{dA_2}{dt} &= c_2 A_2 + \boxed{\lambda_{21} A_2 |A_1|^2} + \lambda_{22} A_2 |A_2|^2 + \boxed{\lambda_{23} A_1^2} \end{aligned}$$

Coupled Landau Equation
"mean flow" resonance

$$\begin{aligned} \frac{dA_1}{dt} &= c_1 A_1 + \lambda_{11} A_1 |A_1|^2 + \boxed{\lambda_{12} A_1 A_2^2} + \boxed{\lambda_{13} A_1 A_2} \\ \frac{dA_2}{dt} &= c_2 A_2 + \boxed{\lambda_{21} A_2 |A_1|^2} + \lambda_{22} A_2^3 + \boxed{\lambda_{23} |A_1|^2} + \boxed{\lambda_{24} A_2^2} \end{aligned}$$

Numerical results awaited

Shukla & Alam (2011) (Preprint)

Conclusions

- Coupled Landau equations have been derived for both cases: resonating mode interaction and non-resonating mode interaction.

- Analytical solutions for the coefficients of coupled Landau equations have been derived for the gradient-banding problem (first problem of the talk).

- Detailed numerical results awaited.

Shukla and Alam (preprint 2011)

Theory for Spatially Modulated Patterns

Complex Ginzburg Landau Equation (CGLE)

Landau Equation

$$\frac{dA}{dt} = c^{(0)}A + c^{(2)}A\,|\,A\,|^2$$

Ordinary differential equation

Holds for **spatially periodic** patterns

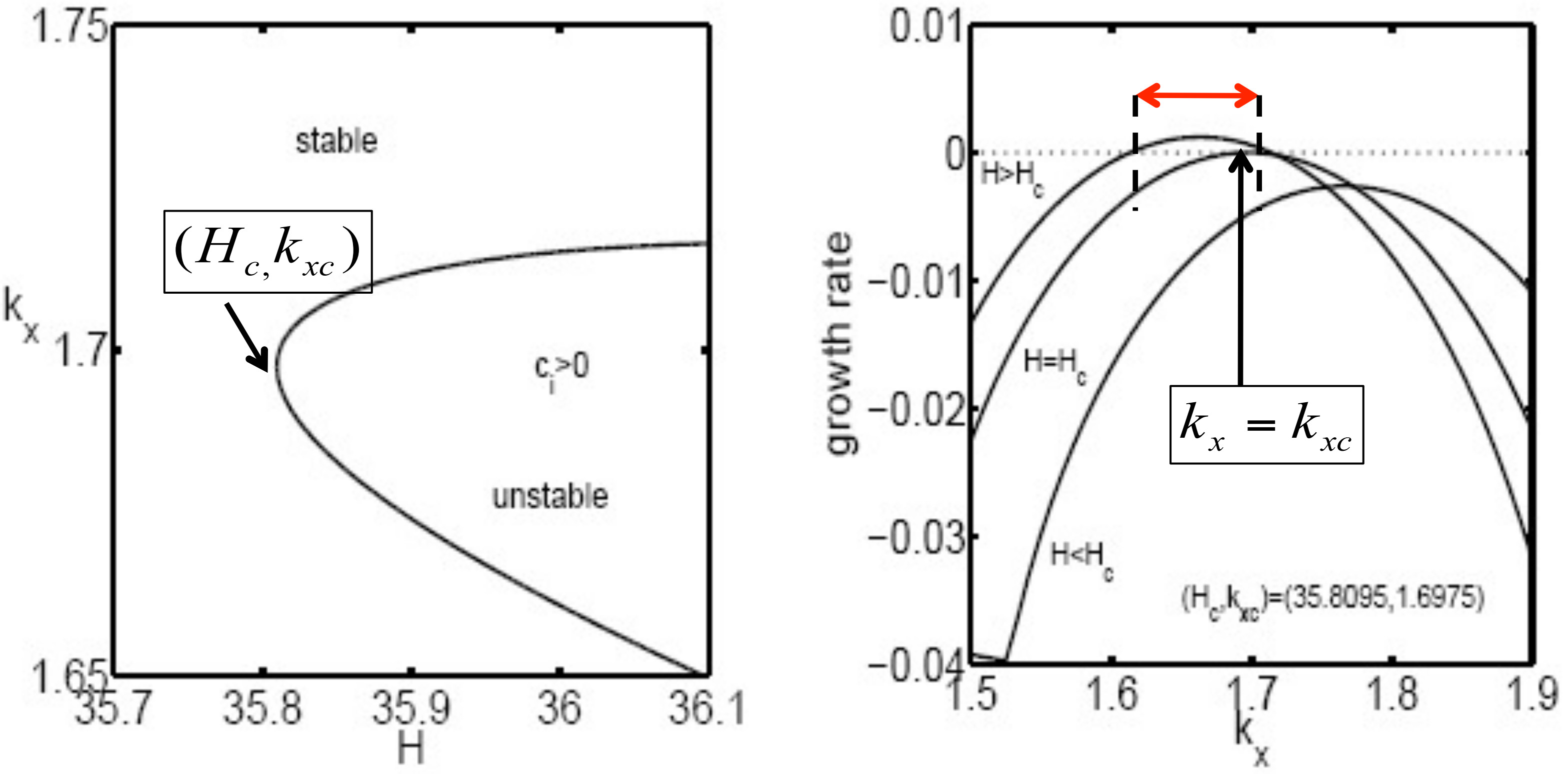
Complex Ginzburg Landau Equation

$$\frac{\partial A}{\partial t} = \varepsilon^2 A + a_2 \frac{\partial^2 A}{\partial X^2} + c^{(2)}A\,|\,A\,|^2$$

Partial differential equation

Holds for spatially modulated patterns

Under which condition CGLE arises?



For $\boxed{H < H_c}$ all modes are decaying : Homogeneous state is stable,

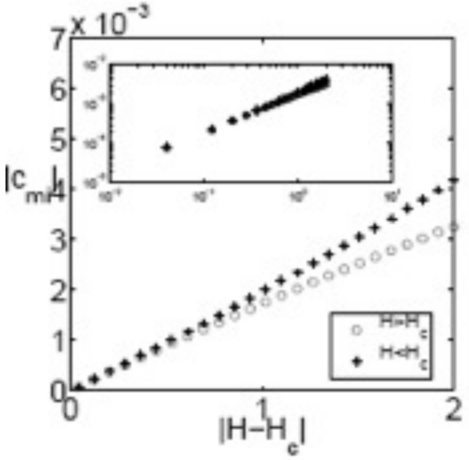
$\boxed{H = H_c}$ at $k_x = k_{xc}$ a critical wave number gains neutral stability,

$\boxed{H > H_c}$ there is a narrow band of wavenumbers around the critical value where the growth rate is slightly positive.

width of the unstable wavenumbers: $\boxed{\propto (H - H_c)^{1/2}}$

Theory (Multiple scale analysis)

$$\left(\frac{\partial}{\partial t} - L\right) X'(x,y,t) = \sum_{i=2}^{\infty} N_i$$



Growth rate is of order $H - H_c$ Stewartson & Stuart (1971)

The timescale at which nonlinear interaction affects the evolution of fundamental mode is of order 1/(growth rate)

$$\varepsilon^2 = d_{1r} \,|\, H - H_c \,|$$

$$\tau = \varepsilon^2 t \longrightarrow \text{Slow time scale}$$

$$\xi = \varepsilon (x - c_g t) \longrightarrow \text{Slow length scale}$$

$$\downarrow$$

Group velocity

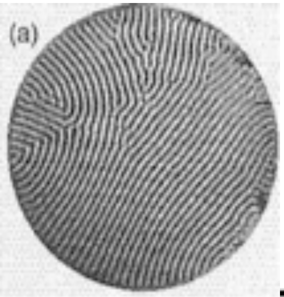
$$\begin{aligned}(\omega I - L_{k_{xc}}) X_{13} \;&=\; \frac{1}{d_{1r}} \frac{\partial L_{k_x}}{\partial H} A X_1 - \frac{\partial A}{\partial \tau} X_1 + G_{13} |A|^2 A \\ &+ \frac{\partial^2 A}{\partial \xi^2} \left[\left(c_g + \frac{1}{i} \left[\frac{\partial L_{k_x}}{\partial k_x} \right] \right) X^{[1;2]} - \frac{1}{2} \left[\frac{\partial^2 L_{k_x}}{\partial k_x^2} \right] X_1 \right]\end{aligned}$$

$$\frac{\partial A(\xi,\tau)}{\partial \tau} \;=\; \frac{d_1}{d_{1r}} A + a_2 \frac{\partial^2 A}{\partial \xi^2} + c^{(2)} |A|^2 A$$

$$\boxed{\frac{\partial A}{\partial t} = \varepsilon^2 A + a_2 \frac{\partial^2 A}{\partial X^2} + c^{(2)} A \,|\, A \,|^2}$$

$$X = x - c_g t$$

Patterns in Vibrated Bed



Patterns in Vibrated bed can be predicted by the complex Ginzburg LE
(Tsimring and Aranson 1997, Blair et. al. 2000)

$$\frac{\partial \psi}{\partial t} = \gamma \psi^* - (1 - i\omega)\psi + (1 + ib)\nabla^2 \psi - |\psi|^2 \psi - \rho \psi$$

Recent work of Saitoh and Hayakawa (Granular Matter 2011) on CGLE in \`\`*unbounded*'' shear flow.

Conclusions

➢Complex Ginzburg Landau equation has been derived that describes spatio-temporal patterns in a \`\`*bounded*'' sheared granular fluid.

➢ Numerical results awaited...

Summary

- Landau-type order parameter theory for the gradient banding in gPCF has been developed using center manifold reduction. **Ref: PRL, vol. 103, 068001, (2009)**

- Analytical solution for the shearbanding instability, comparison with numerics & bifurcation scenario have been unveiled. **Ref: JFM, vol. 666, 204-253, (2011a)**

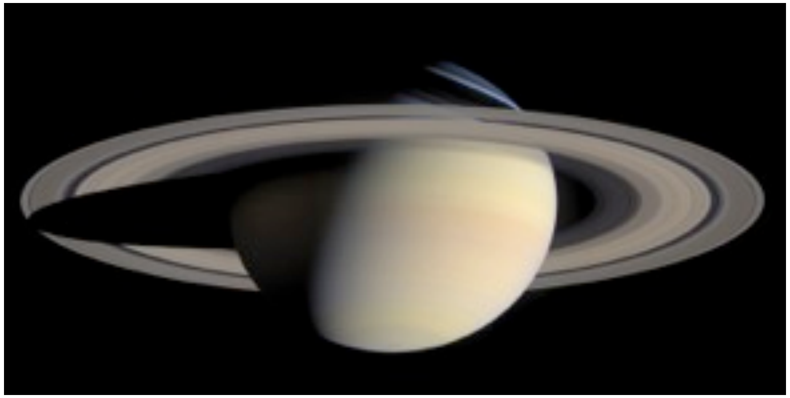
- The order parameter theory for 2D-gPCF has been developed. Nonlinear patterns and bifurcations have been studied. **Ref: JFM, vol. 672, 147-195 (2011b)**

- Nonlinear analysis for the gradient and vorticity banding in 3D-GPCF has been carried out. **Submitted (2011c)**

- Coupled Landau equations for resonating and non-resonating cases have been derived. **Preprint**

- Complex Ginzburg Landau equation has been derived for bounded shear flow. **Preprint**

Revisit nonlinear theory of Saturn's Ring



- ➢Non-isothermal model with spin, stress anisotropy... Text
- ➢ Self-gravity, Corriolis and Tidal forces ...??
- ➢ Spatially modulated waves (Joe's talk)...
- ➢ Wave interactions (Jurgen's comment)...
- ➢ Secondary instability, ...

THANK YOU