Nonequilibrium Dynamics in Astrophysics and Material Science
2011-11-02 @ YITP, Kyoto

Multi-scale coherent structures and their role in the Richardson cascade of turbulence

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1.

Background

A key issue

Small-scale universality of turbulence statistics

Kolmogorov (1941), Dokl. Akad. Nauk SSSR, **30**.
English translation: Proc. R. Soc. Lond. A **434** (1991) 9-13.

The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers†

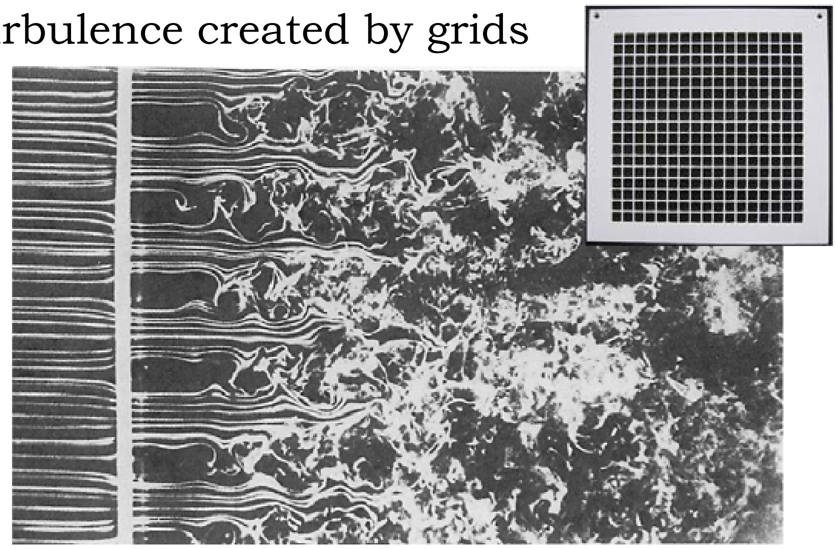
Similarity hypothesis:

Small-scale statistics do not depend on the b.c. or the forcing sustaining turbulence.

domain G of the four-dimensional space (x_1, x_2, x_3, t) are *random variables* in the sense of the theory of probabilities (cf. for this approach to the problem Millionshtchikov (1939)).

Small-scale universality

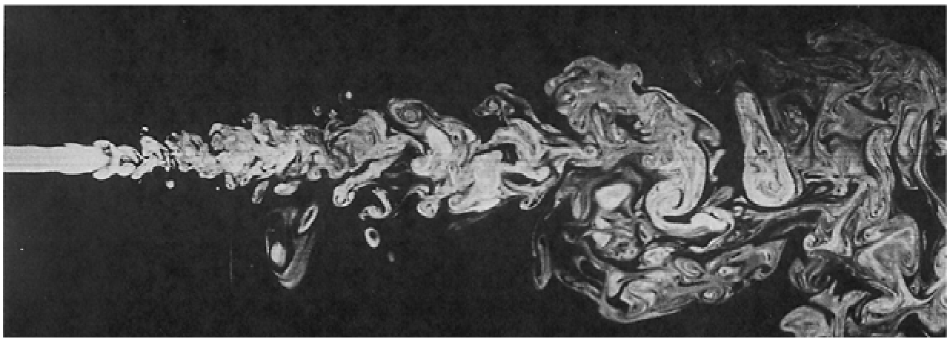
Turbulence created by grids



Van Dyke, An album of fluid motion

Small-scale universality

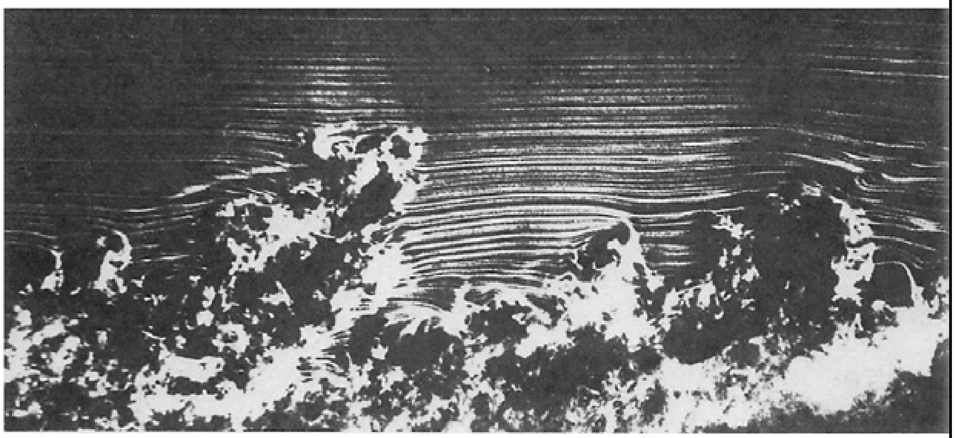
Turbulent jet



Van Dyke, An album of fluid motion

Small-scale universality

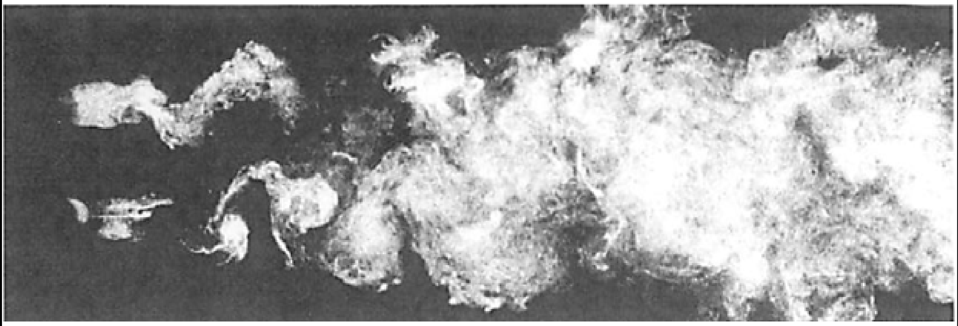
Turbulence in a boundary layer



Van Dyke, An album of fluid motion

Small-scale universality

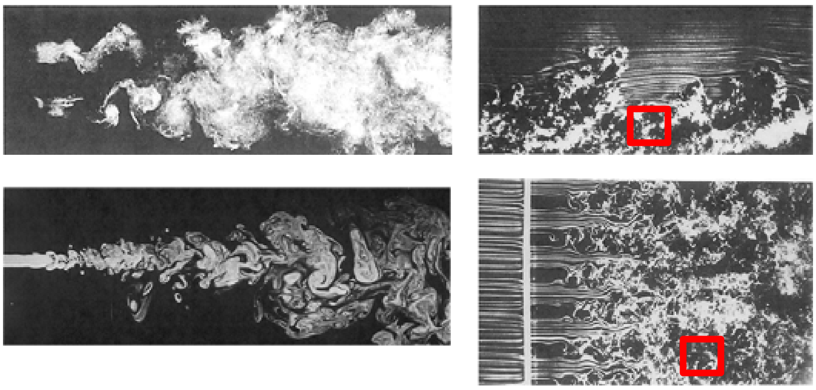
Turbulent wake behind (a pair of) cylinders



Van Dyke, An album of fluid motion

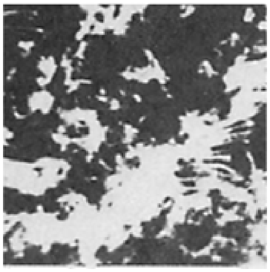
Small-scale universality

Small eddies far from boundaries look similar....

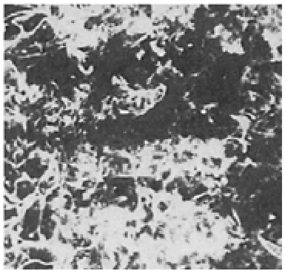


Van Dyke, An album of fluid motion

Small-scale universality



Boundary layer turbulence



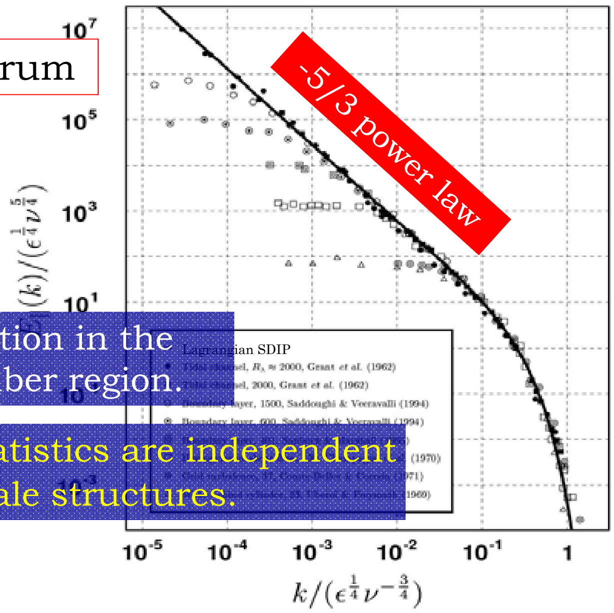
Turbulence behind grids

Kolmogorov (1941) Their statistics are independent of b.c., and determined by

- **the energy dissipation rate** ε &
- **the kinematic viscosity** ν.

Small-scale universality: an evidence

Energy spectrum



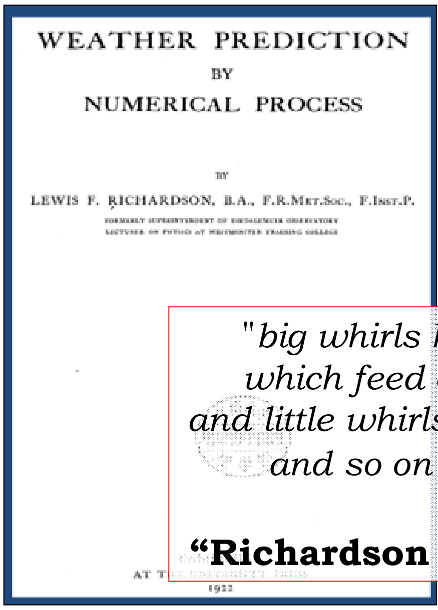
Universal function in the high-wavenumber region.

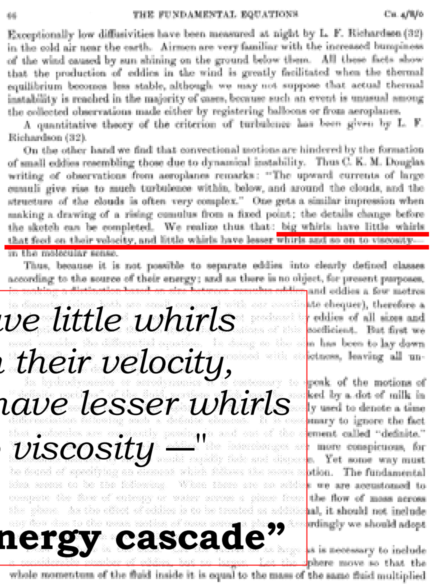
Small-scale statistics are independent of the large-scale structures.

Why?

Because of the <u>cascade</u> of energy….

Richardson 1922





THE FUNDAMENTAL EQUATIONS

Exceptionally low diffusivities have been measured at night by L. F. Richardson (32) in the cold air near the earth. Airmen are very familiar with the increased bumpiness of the wind caused by sun shining on the ground below them. All these facts show that the production of eddies in the wind is greatly facilitated when the thermal equilibrium becomes less stable, although we may not suppose that actual thermal instability is reached in the majority of cases, because such an event is unusual among the collected observations made either by registering balloons or from aeroplanes.

A quantitative theory of the criterion of turbulence has been given by L. F. Richardson (32).

On the other hand we find that convectional motions are hindered by the formation of small eddies resembling those due to dynamical instability. Thus C. K. M. Douglas writing of observations from aeroplanes remarks: "The upward currents of large cumuli give rise to much turbulence within, below, and around the clouds, and the structure of the clouds is often very complex." One gets a similar impression when making a drawing of a rising cumulus from a fixed point; the details change before the sketch can be completed. We realize thus that: big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls and so on to viscosity—in the molecular sense.

Thus, because it is not possible to separate eddies into clearly defined classes according to the source of their energy; and as there is no object, for present purposes, ...

"big whirls have little whirls
which feed on their velocity,
and little whirls have lesser whirls
and so on to viscosity —"

"Richardson energy cascade"

Frisch’s schematic picture

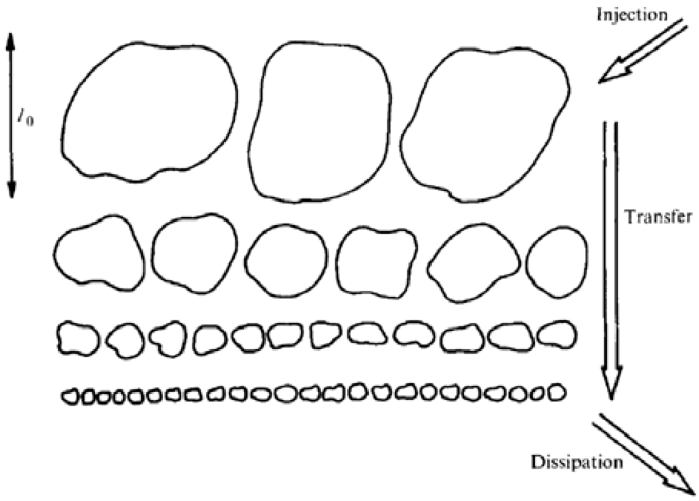


FIGURE 1. The energy cascade according to the 1941 Kolmogorov theory.

Frisch, Sulem & Nelkin, Journal of Fluid Mech. (1978).

Frisch’s schematic picture

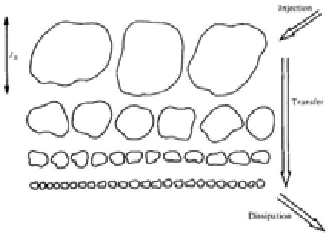


FIGURE 1. The energy cascade according to the 1941 Kolmogorov theory.

- ◆**Energy is injected at a large scale,**
- ◆**transferred to smaller scales (scale-by-scale),**
- ◆**dissipated at the very small scale.**

Through this scale-by-scale energy cascade the information of the b.c./forcing is lost.

Question

What is the mechanism of the energy cascade in turbulence?

Wave-number space analyses based on numerical simulations have been done to verify the cascade picture:

e.g.
Domaradzki, J. A. & Rogallo, R. S., Phys. Fluids A **2** (1990) 413-426.
Yeung, P. K. & Brasseur, J. G., Physics of Fluids A **3** (1991) 884-897.
Ohkitani, K. & Kida, S., Physics of Fluids A **4** (1992) 794-802.

Several verses proposed…

Richardson (1922)

"big whirls have little whirls
which feed on their velocity,
and little whirls have lesser whirls
and so on to viscosity —"

Betchov (cited by Tsinober 1991)

"Big whirls lack smaller whirls
to feed on their velocity,
they crash and form the finest curls,
permitted by viscosity—"

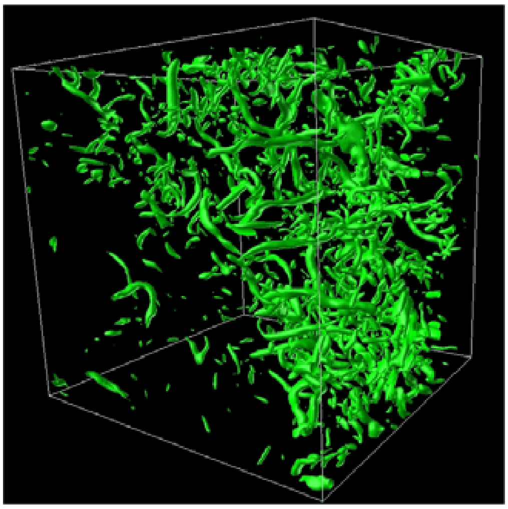
Hunt (2010)

"Great whirls gobble smaller whirls
and feed on their velocity: but
where great whirls grind, they also slow,
and little whirls begin to grow
— stretching out with high vorticity"

….

… unsolved problem …

Clue



Turbulence is not random, but consists of coherent structures.

Aim

to describe the physical mechanism of Richardson energy cascade

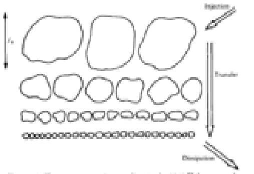
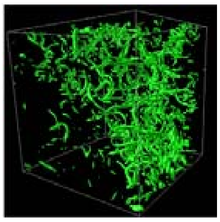


FIGURE 1. The energy cascade according to the 1941 Kolmogorov theory.

in terms of coherent structures:



NB: Characteristic length scales

◆ The length scale of the **largest** eddies:
Integral length L

◆ The length scale of the **smallest** eddies:
Kolmogorov length η

Between L and η, turbulence does not have any characteristic length scale, and it is **statistically self-similar**.

"Inertial range"

Reynolds # = width of inertial range

◆ The length scale of the **largest** eddies:
Integral length L

◆ The length scale of the **smallest** eddies:
Kolmogorov length η

$$\frac{L}{\eta} \sim Re^{\frac{3}{4}}$$

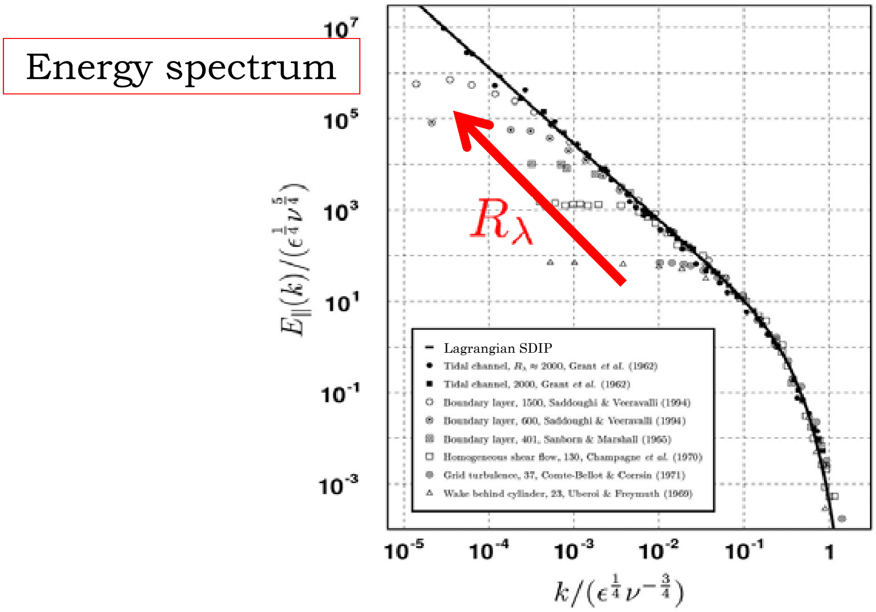
◆ **Reynolds number**

$$Re = \frac{u'L}{\nu}$$

◆ **Taylor-length Reynolds #**

$$R_\lambda \sim \sqrt{Re}$$

Small-scale universality: an evidence



Example of huge-Re turbulence

Tsuji & Dhruva (1999) Physics of Fluids
"Intermittency feature of shear stress fluctuation in high-Reynolds-number turbulence"

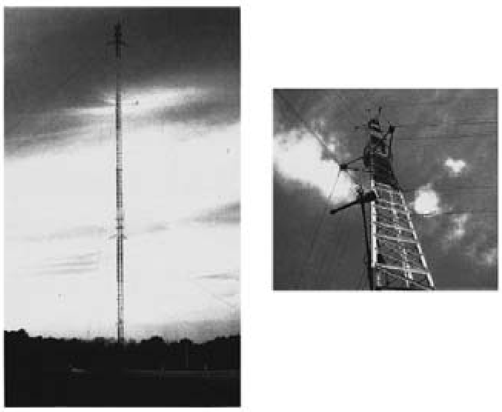
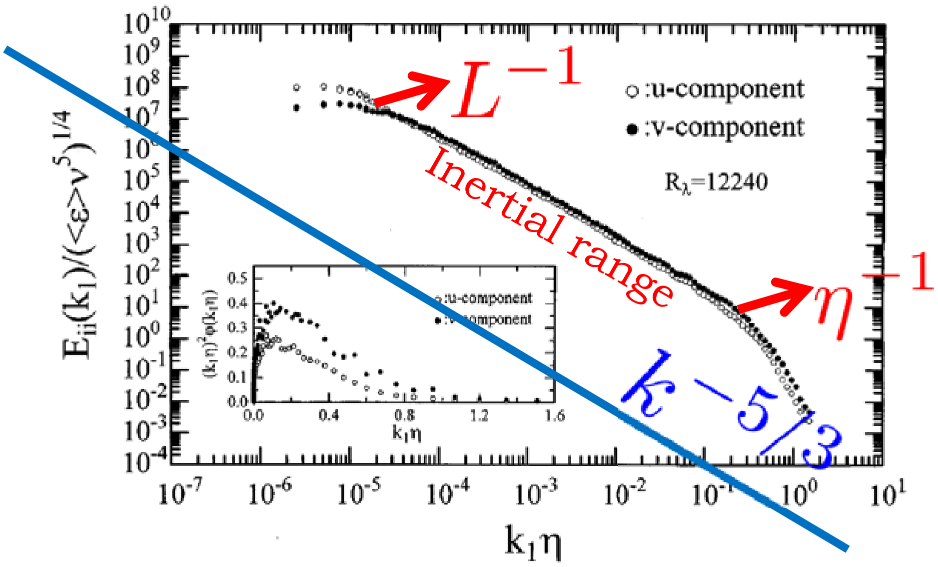


FIG. 1. The observation tower in Brookhaven National Laboratory. The data were obtained at the station 35 m above the ground level.

Energy spectrum & L, η

Tsuji & Dhruva (1999) Physics of Fluids



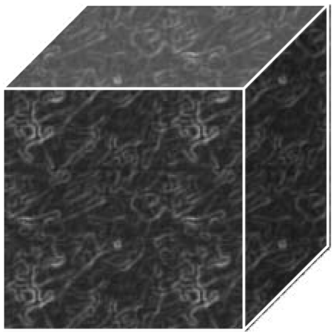
2.

Coherent structures in turbulence

(numerical simulation)

Turbulence in periodic cube

Need to simulate turbulence at Reynolds numbers as high as possible.



Periodic boundary conditions in all the three orthogonal directions.

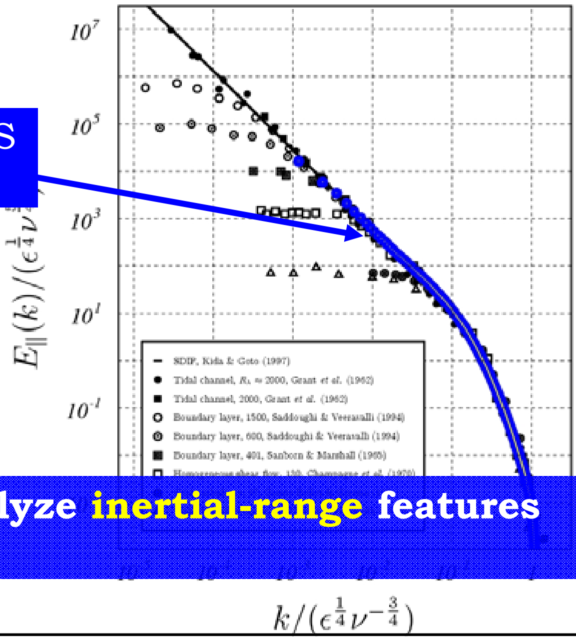
Model of (small-scale) turbulence far from walls.

Numerical scheme

- Direct integration of the Navier-Stokes eq. (4^{th} order Runge-Kutta scheme)
- Incompressible, Newtonian fluid
- Artificial forcing at large scales
- Statistically homogeneous/isotropic/steady
- Fourier spectral method
- 2048^3 grid points
- **Taylor-length Reynolds number $R_\lambda = 540$**

Energy spectrum

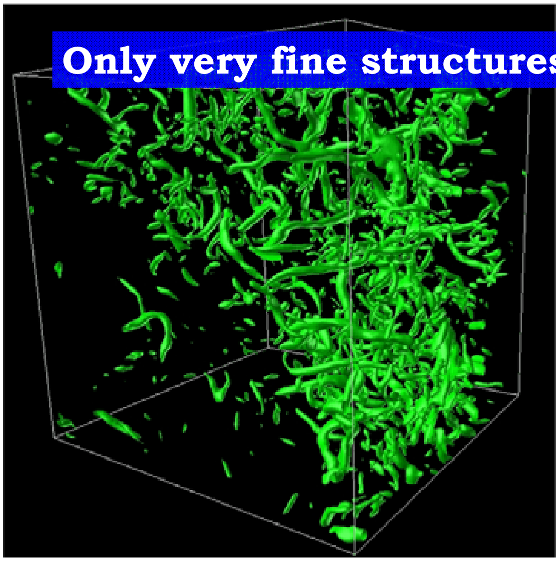
the current DNS ($R_\lambda = 540$)



Ready to analyze inertial-range features by DNS.

Iso-surfaces of enstrophy (ω^2)

Only very fine structures are observed.



$$|\omega| = |\nabla \times \boldsymbol{u}|$$

side = 1300η

$(1/4)^3$ of the box

Coarse-grained enstrophy

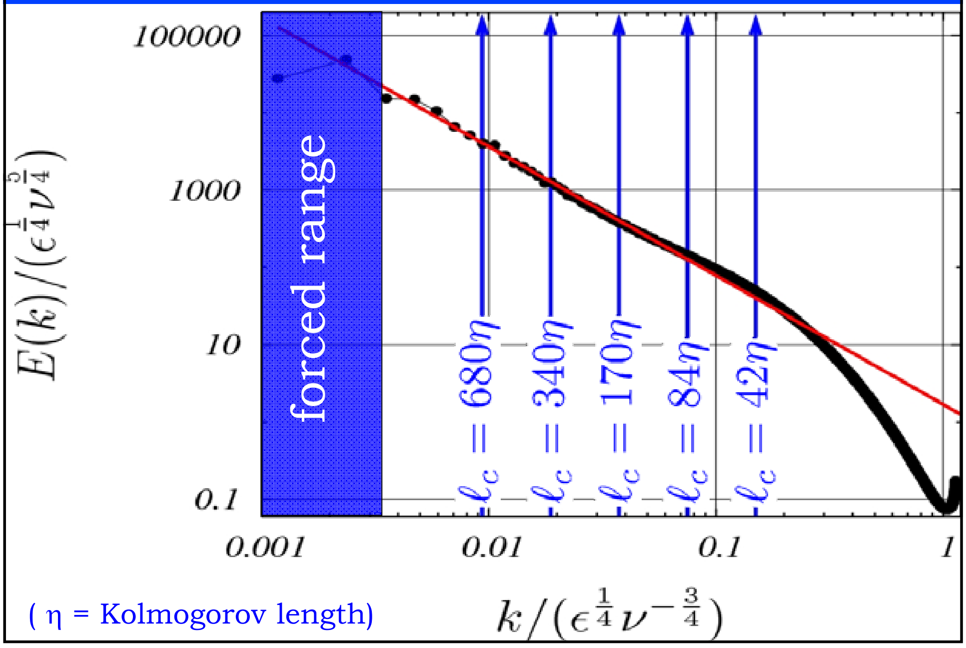
To identify the coherent structures in the inertial range...

The simplest method:

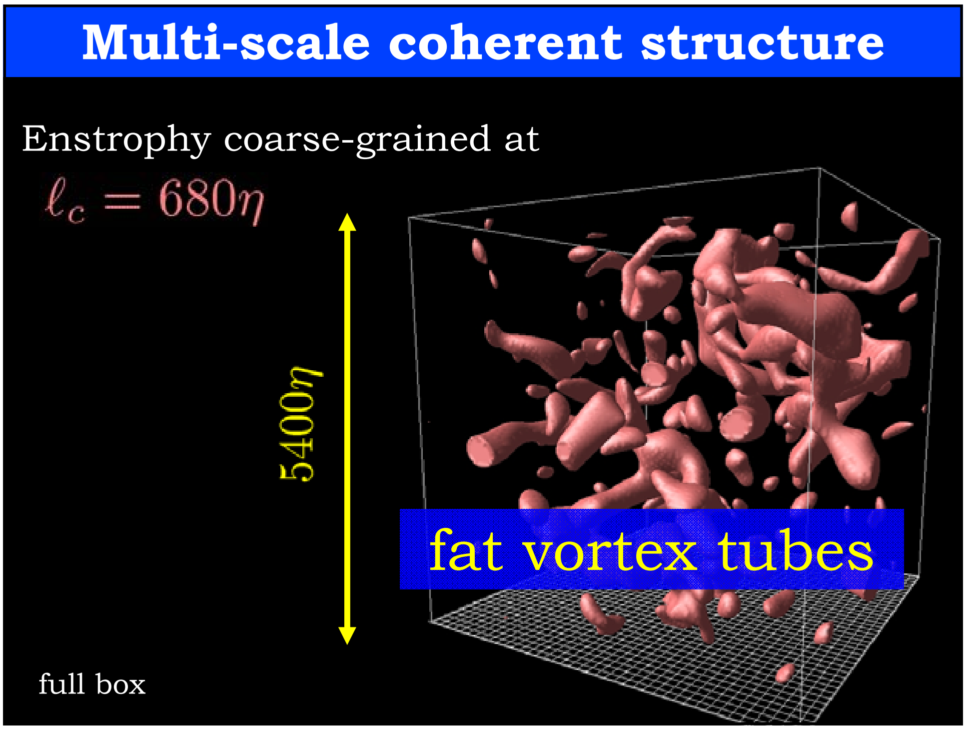
Coarse-graining the velocity gradients
by the low-pass filtering of the Fourier components.

→ Iso-surfaces of
the coase-grained enstrophy or strain rate.

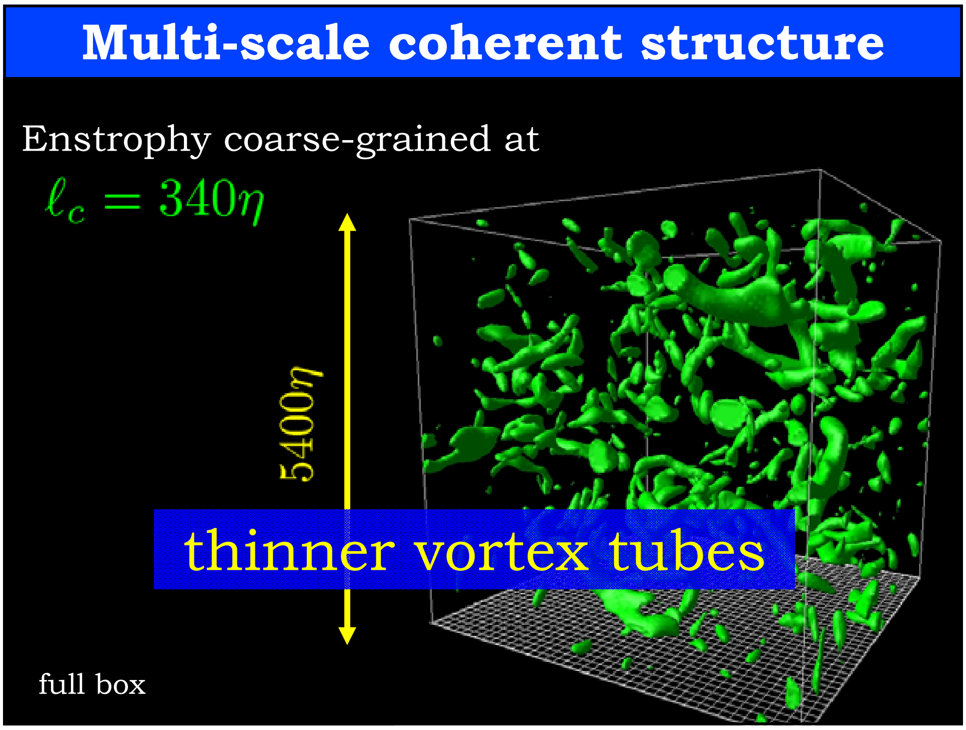
Coarse-graining scales



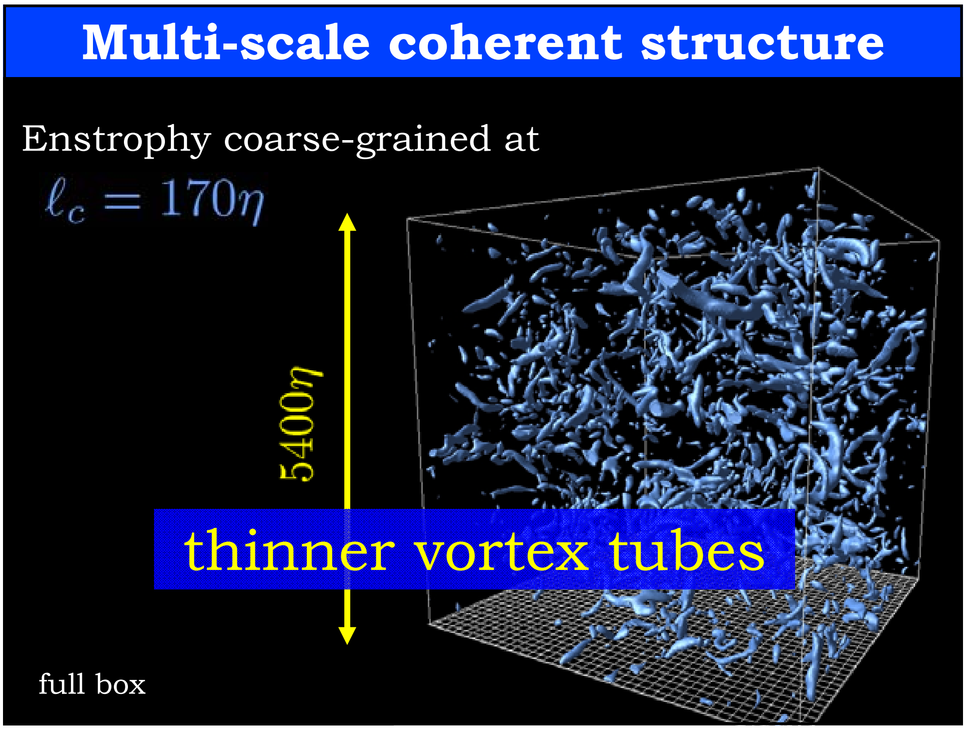
(η = Kolmogorov length)

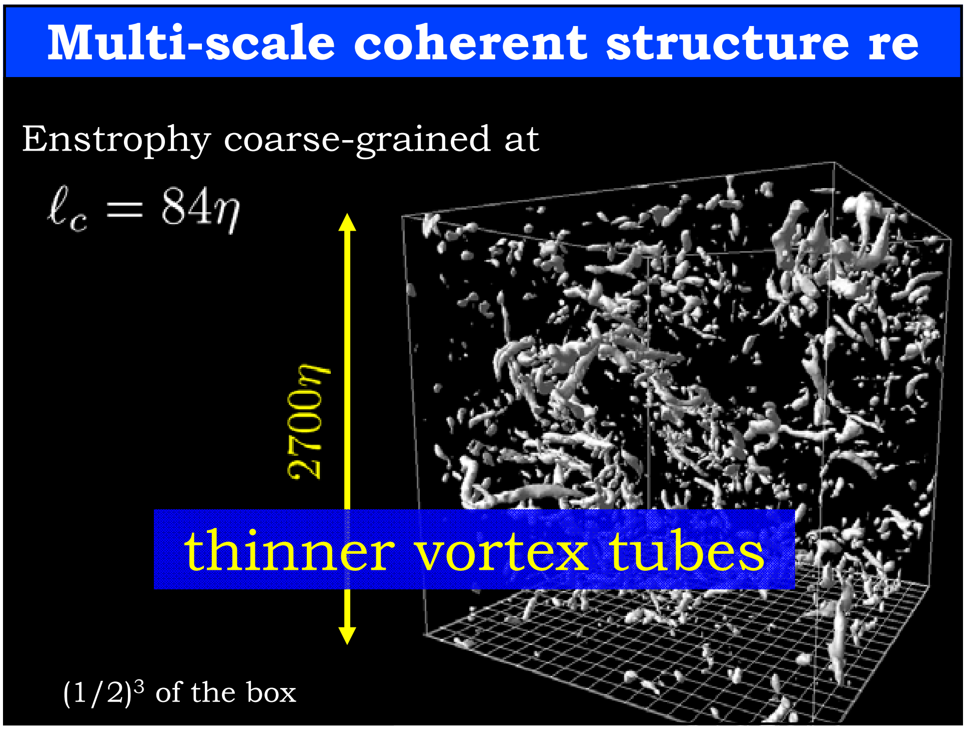


Multi-scale coherent structure
Enstrophy coarse-grained at
$\ell_c = 680\eta$
5400η
fat vortex tubes
full box



Multi-scale coherent structure
Enstrophy coarse-grained at
$\ell_c = 340\eta$
5400η
thinner vortex tubes
full box


Multi-scale coherent structure
Enstrophy coarse-grained at
$\ell_c = 170\eta$
5400η
thinner vortex tubes
full box

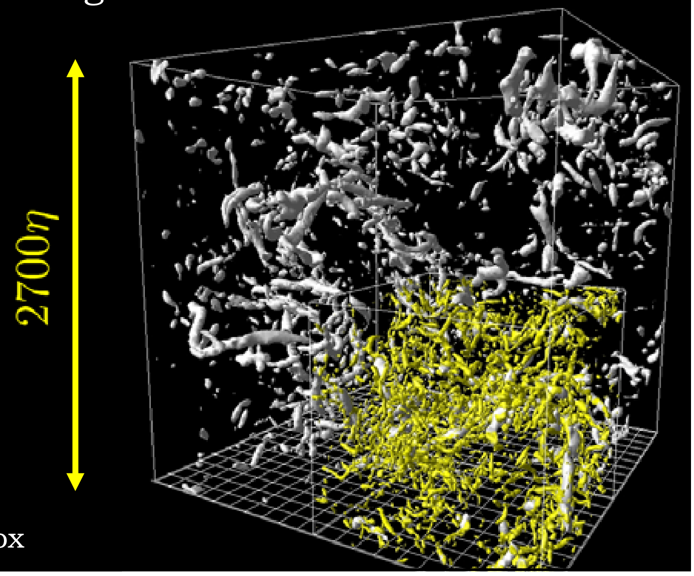

Multi-scale coherent structure re
Enstrophy coarse-grained at
$\ell_c = 84\eta$
2700η
thinner vortex tubes
$(1/2)^3$ of the box

Multi-scale coherent structure

Enstrophy coarse-grained at

$\ell_c = 84\eta$

$\ell_c = 42\eta$

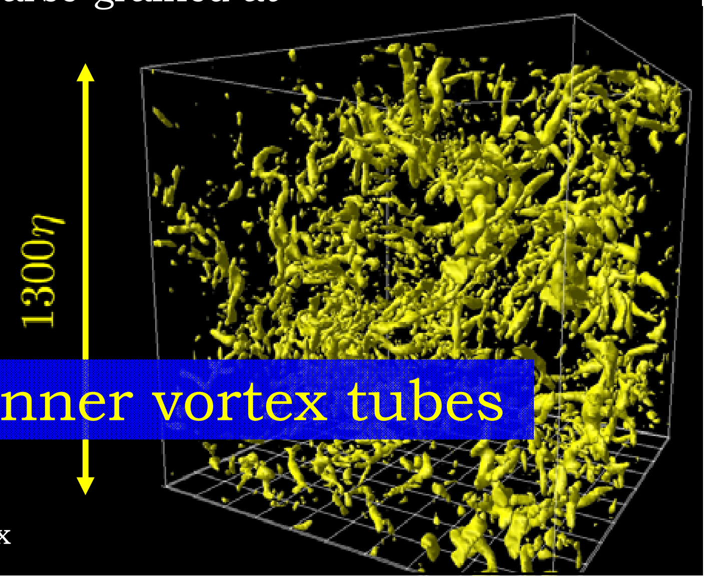


$(1/2)^3$ of the box

Multi-scale coherent structure

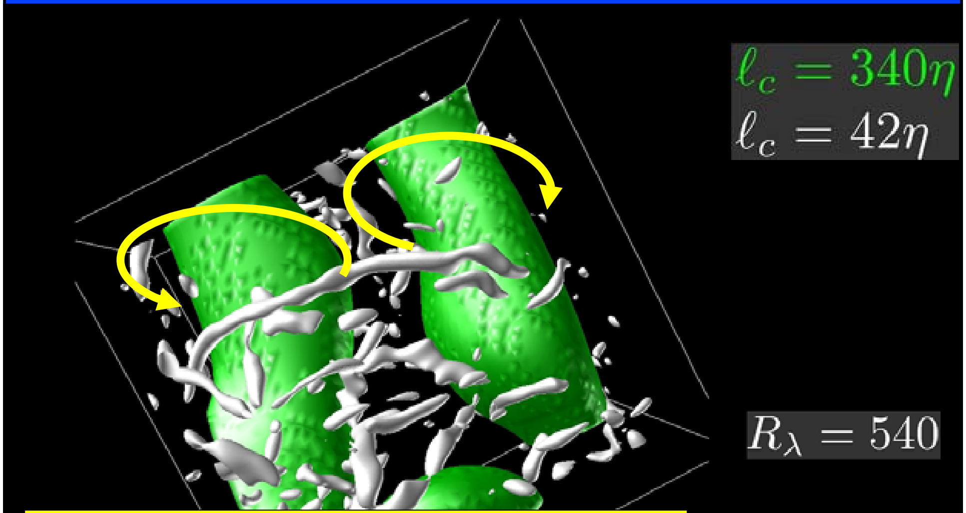
Enstrophy coarse-grained at

$\ell_c = 42\eta$



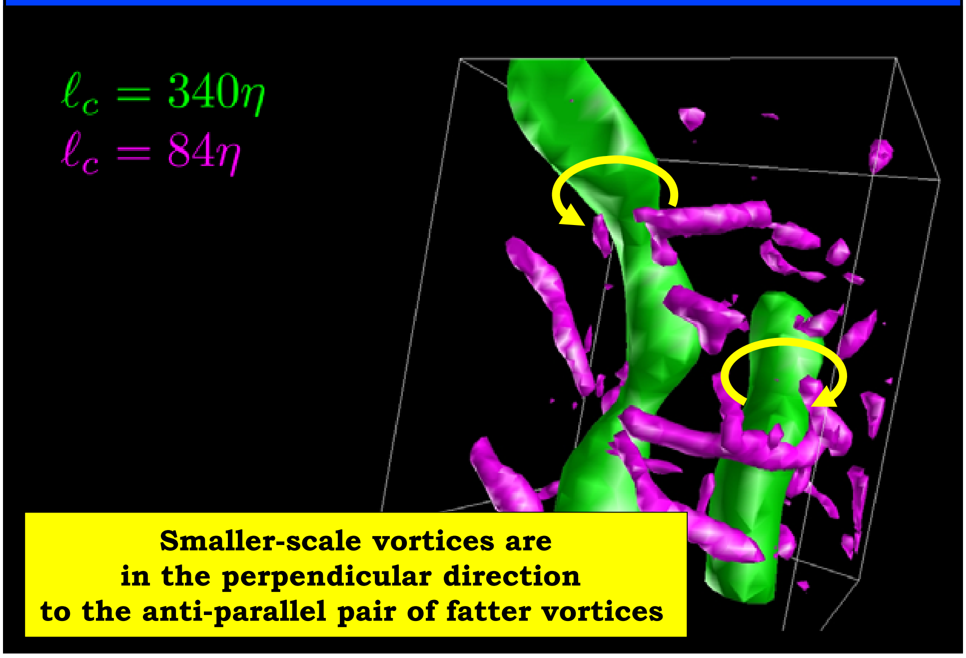
$(1/4)^3$ of the box

CS at different scales



Smaller-scale vortices are in the perpendicular direction to the anti-parallel pair of fatter vortices

Another example



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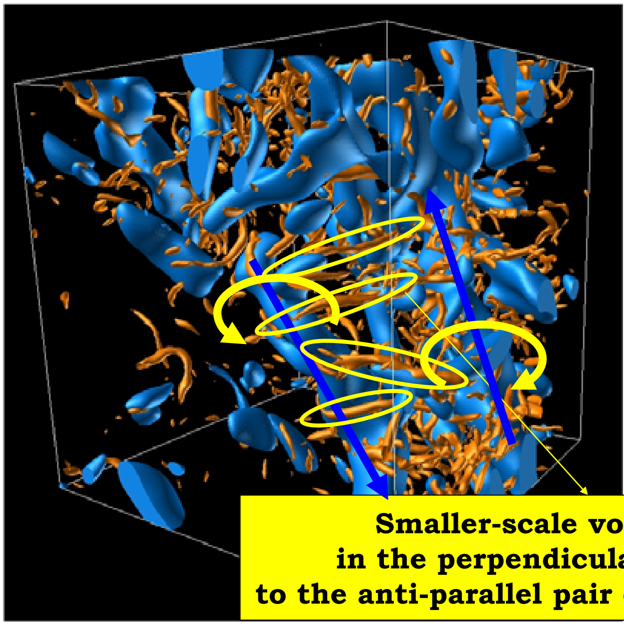


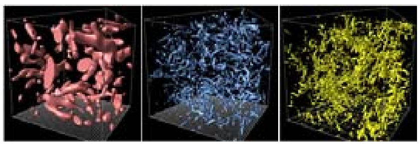
$\ell_c = 84\eta$
$\ell_c = 21\eta$

Smaller-scale vortices are in the perpendicular direction to the anti-parallel pair of fatter vortices

DNS observations

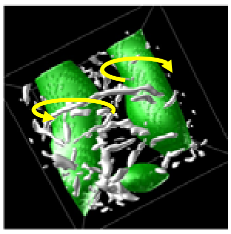
At a scale in the inertial range:

Coherent vortices have tubular shapes,
whose radii are comparable to the scale.

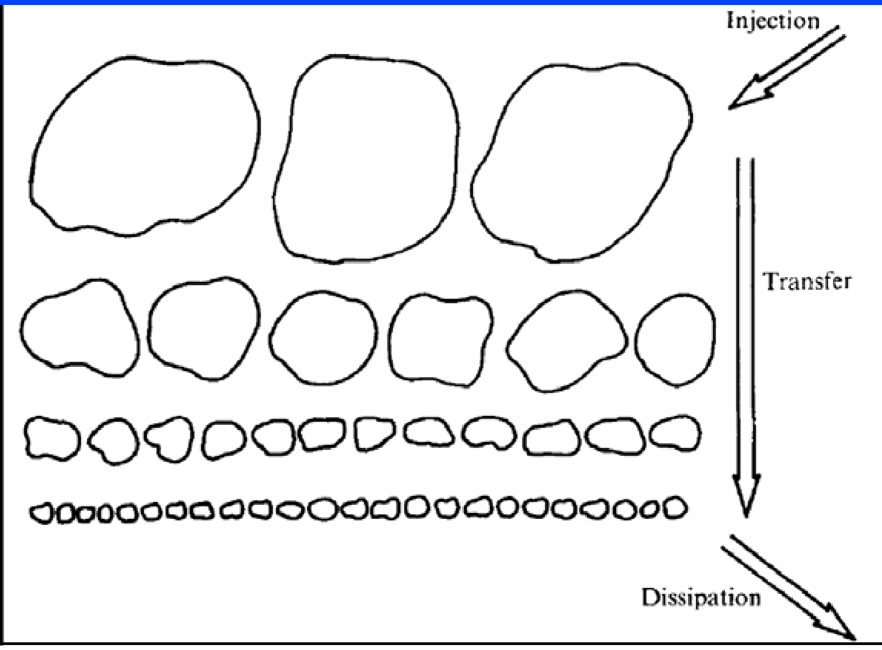


At different scales:

Smaller-scale tubes tend to align in the perpendicular direction to (the anti-parallel pairs of) larger-scale tubes.



Energy cascade



3. (supplement)

Vortex dynamics

Vorticity equation

$$\boldsymbol{\omega} \equiv \boldsymbol{\nabla} \times \boldsymbol{u}$$

($\boldsymbol{u}$ = velocity)

$\boldsymbol{\nabla} \times$ (N.-S. eq.)

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega} \cdot \boldsymbol{\nabla}\, \boldsymbol{u} + \nu\, \nabla^2\, \boldsymbol{\omega}$$

Similar to the magnetic field…

- ◆ In the limit of zero viscosity, vorticity is frozen in fluid.
- ◆ Vorticity is strengthened by stretching, and weaken by diffusion.

Biot-Savart law

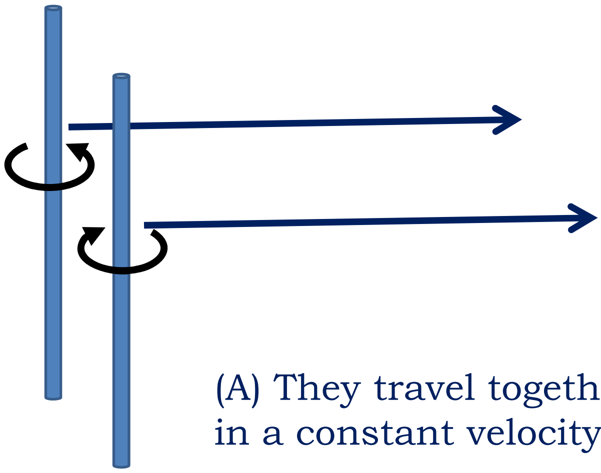
$$\boldsymbol{u} = \frac{1}{4\pi} \int \frac{d\boldsymbol{\omega} \times \boldsymbol{r}}{r^3}$$

- ◆ Velocity is determined by the vorticity field.

→ Vorticity is dynamically important.

Students' exercise

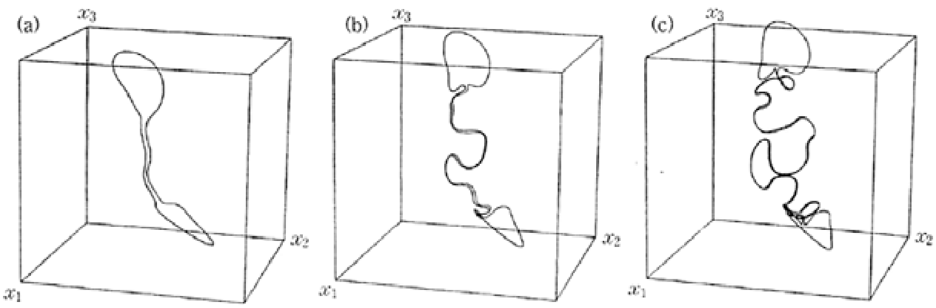
(P) What is the motion of an anti-parallel pair of line vortices with a same circulation?



(A) They travel together in a constant velocity.

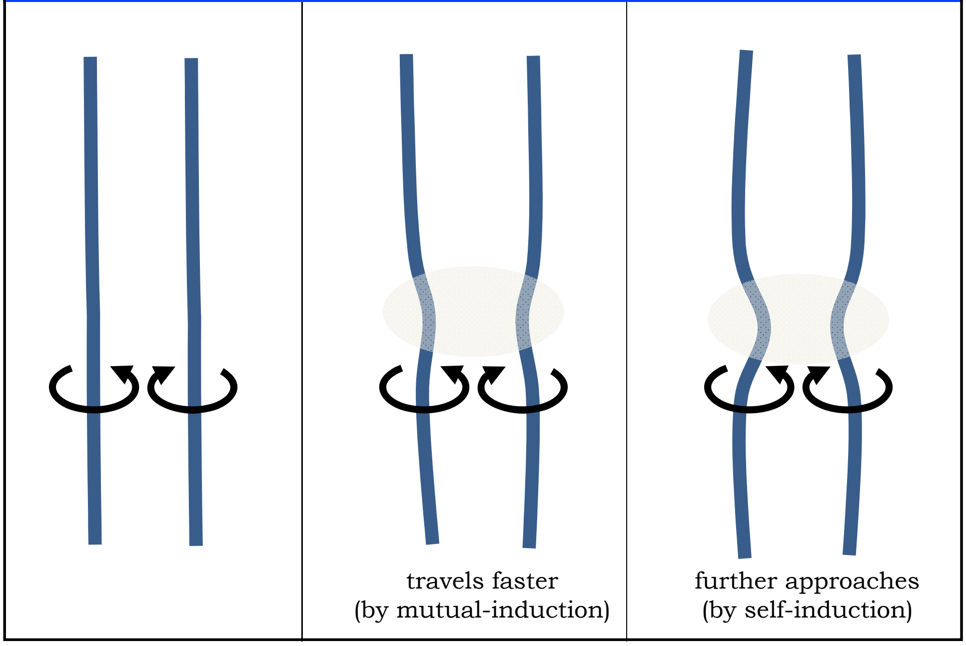
Real answer

They approach to each other.

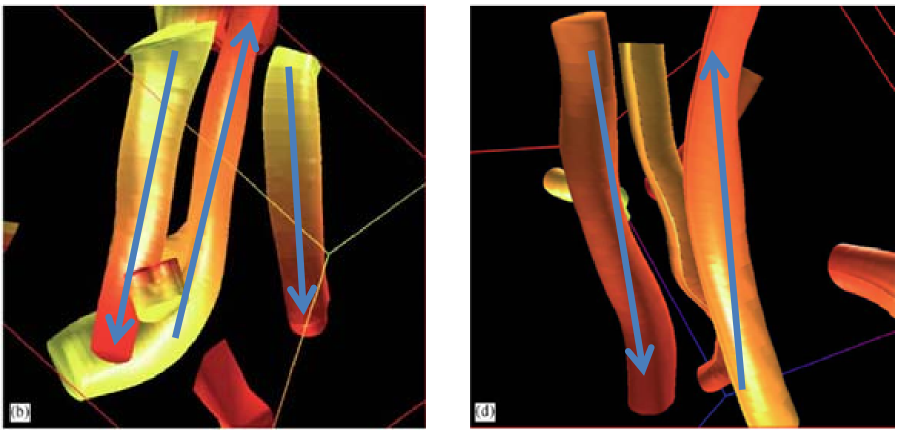


"Collapse and Amplification of a Vortex Filament", E.D. Siggia, Phys. Fluids **28**, 794 (1985)

Siggia's mechanism



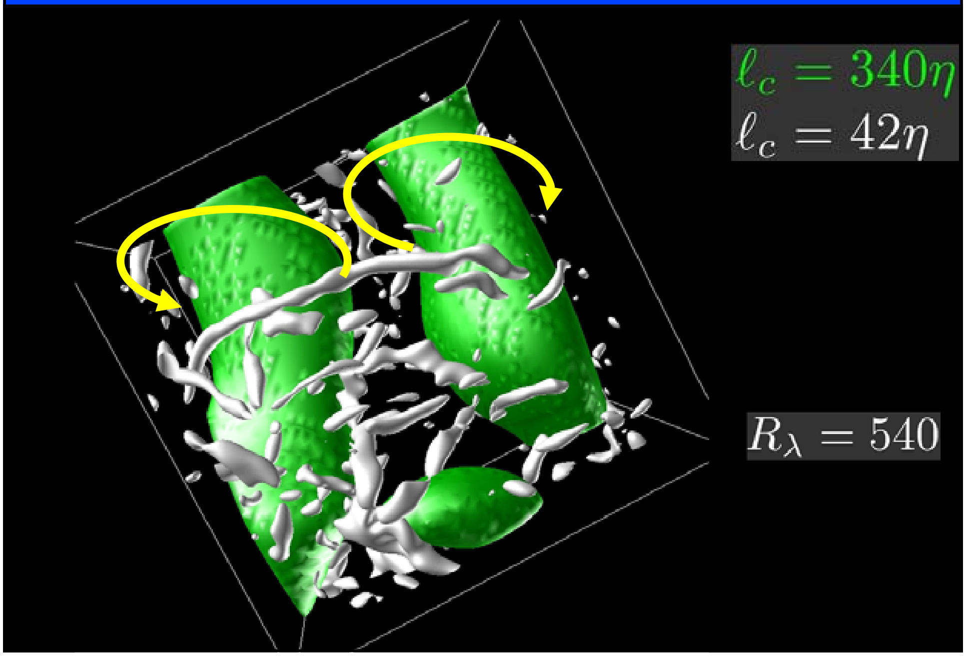
Anti-parallel pairs in turbulence



at the smallest (Kolmogorov) scale.

Goto & Kida, "Enhanced stretching of material lines by antiparallel vortex pairs in turbulence," Fluid Dynamics Research **33** (2003) 403–431.

Anti-parallel pairs in inertial range



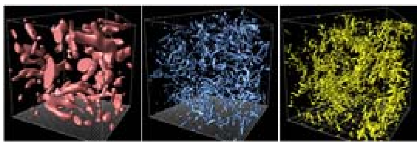
4.

Richardson cascade

Recall: DNS observations

At a scale in the inertial range:

Coherent vortices have tubular shapes, whose radii are comparable to the scale.



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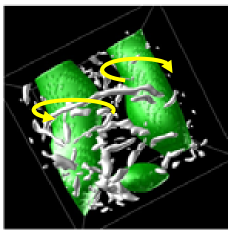
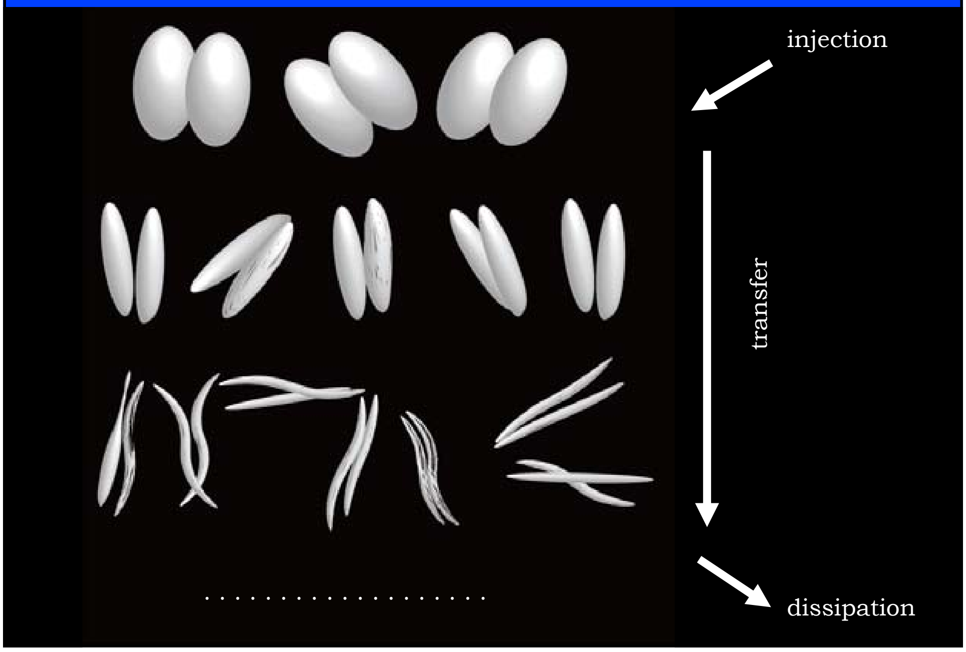


Energy cascade

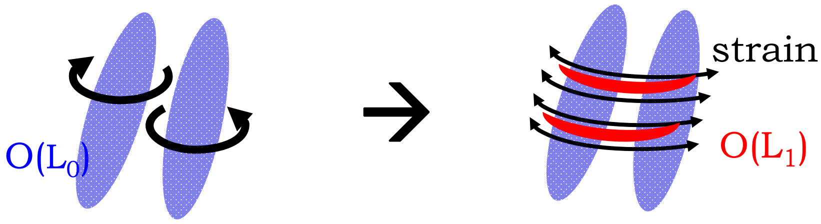


A scenario of the cascade (1/2)

Goto, JFM, **605** (2008) 355.

1. Energy supplied by an external forcing is possessed by the tubular vortices of radius $O(L_0)$.

2. When a pair (especially, anti-parallel pair) of them encounters, smaller-scale ($O(L_0)$, say) vortices are created by vortex-stretching in the strongly straining region around the pair.



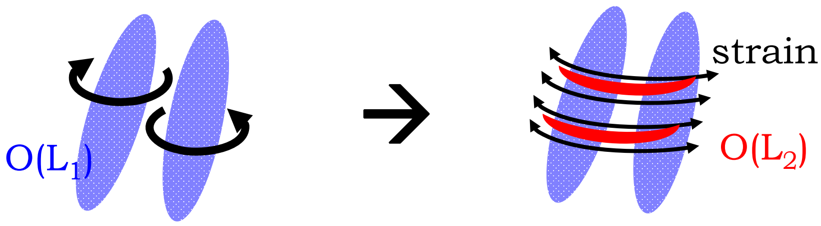
3. Then, the energy transfers from $O(L_0)$ to $O(L_1)$.

A scenario of the cascade (2/2)

Goto, JFM, **605** (2008) 355.

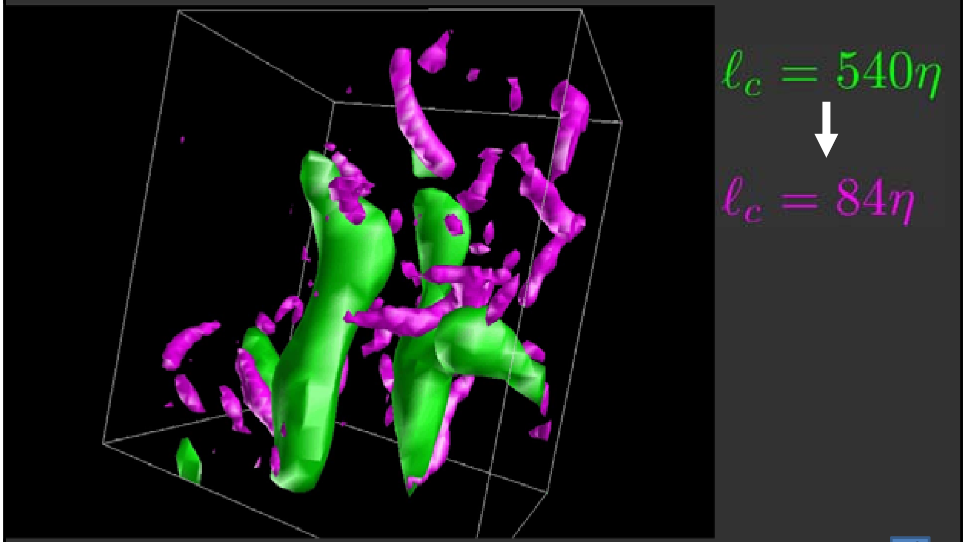
4. The energy transferred to $O(L_1)$ is possessed by tubular vortices of radius $O(L_1)$.

5. When a pair (especially, anti-parallel pair) of them encounters, smaller-scale ($O(L_2)$, say) vortices are created by vortex-stretching in the strongly straining region around the pair.

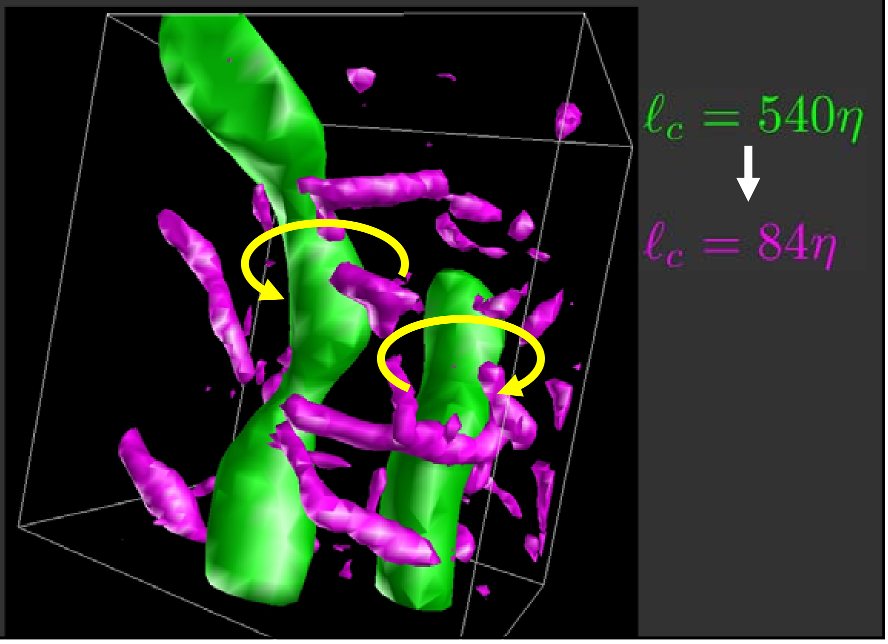


6. Then, the energy transfers from $O(L_1)$ to $O(L_2)$.

An event supporting the scenario



An event supporting the scenario

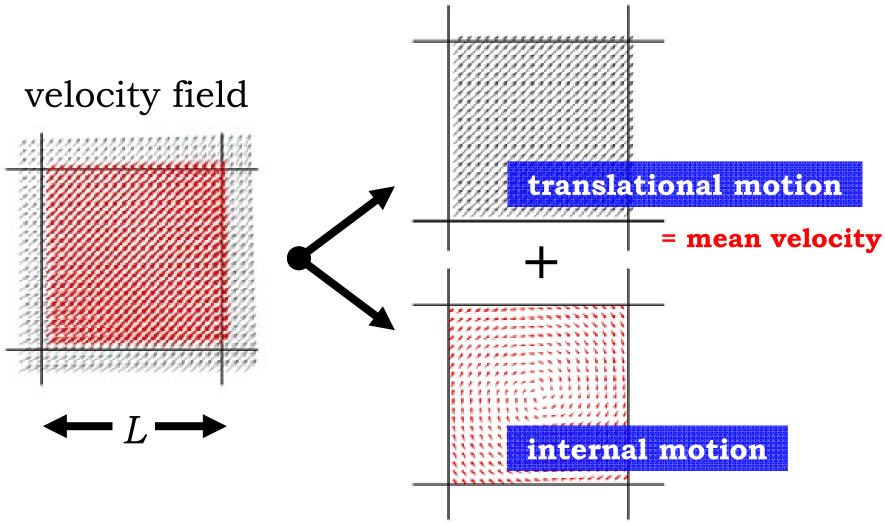


5.

Verification of the scenario

Scale-dependent energy

We define the "internal energy" of fluid particles, as the function of scale L.



Scale-dependent energy

"Internal energy" of scale L at position $\boldsymbol{x}$:

$$U(\boldsymbol{x},t|L) = \frac{1}{2L^3} \int_{V(\boldsymbol{x}|L)} \left|\boldsymbol{u}(\boldsymbol{x}',t) - \langle \boldsymbol{u}(\boldsymbol{x},t)\rangle_L\right|^2 \,\mathrm{d}\boldsymbol{x}'$$

← Energy possessed by structures smaller than L.

Note that U is Galilean invariant.

Scale-dependent energy "transfer"

dU/dt depends on the frame of reference.

→ Choose the frame moving with the translational velocity.

→ The rate of change of energy of each fluid particle in this frame:

$$\boldsymbol{a}(\boldsymbol{x},t) \cdot (\boldsymbol{u}(\boldsymbol{x},t) - \langle \boldsymbol{u}(\boldsymbol{x},t)\rangle_L)$$

acceleration velocity in this frame

Scale-dependent energy "transfer"

At position ***x*** …

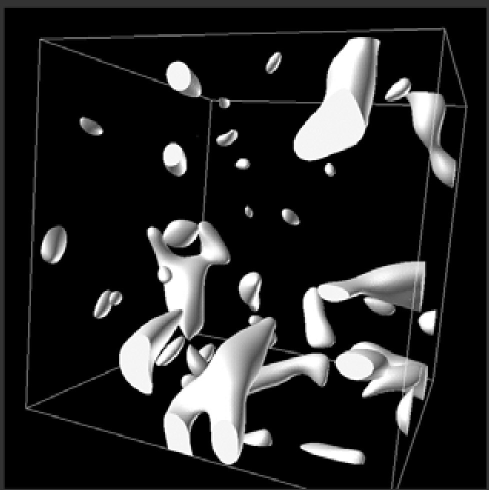
Energy gain of structures smaller than L

$$T(\boldsymbol{x},t|L) = \frac{1}{L^3}\int_{V(\boldsymbol{x}|L)} \boldsymbol{a}(\boldsymbol{x}',t)\cdot\left(\boldsymbol{u}(\boldsymbol{x}',t) - \langle \boldsymbol{u}(\boldsymbol{x},t)\rangle_L\right)\,\mathrm{d}\boldsymbol{x}'$$

Energy gain of each fluid particle in the frame moving with the translational velocity at L.

Note that T is also Galilean invariant.

Vortex tubes



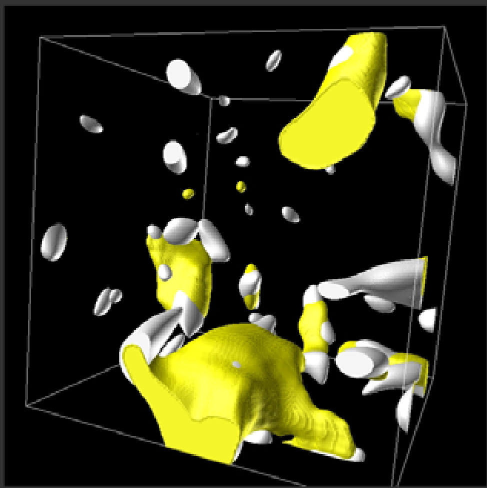
$\ell_c = 170\eta \quad (\in \text{ ISR})$

$(1/2)^3$ of the box

$R_\lambda = 540$

white = coarse-grained enstrophy

Vortices & Energy



$\ell_c = 170\eta \quad (\in \text{ ISR})$

Energy is confined in vortical regions.

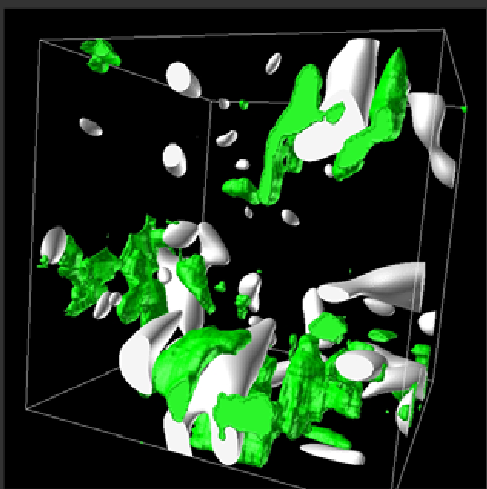
$(1/2)^3$ of the box

$R_\lambda = 540$

white = coarse-grained enstrophy

yellow = energy at this scale

Vortices & Energy transfer



$\ell_c = 170\eta \quad (\in \text{ ISR})$

Energy transfers between vortices.

$(1/2)^3$ of the box

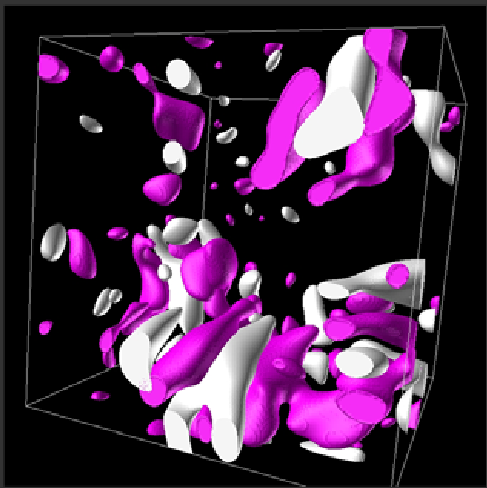
$R_\lambda = 540$

white = coarse-grained enstrophy

green = negative energy transfer

Coarse-grained vorticity & strain

$\ell_c = 170\eta \quad (\in \text{ ISR})$



Energy transfers in straining regions.

$(1/2)^3$ of the box

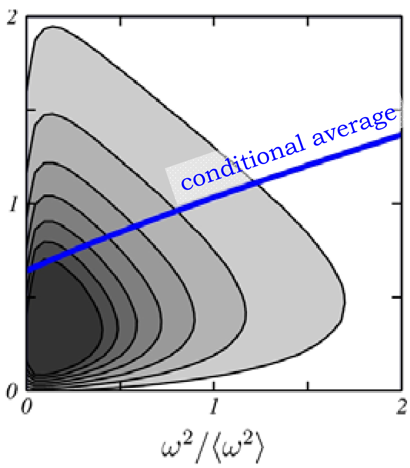
$R_\lambda = 540$

white = coarse-grained enstrophy

purple = coarse-grained strain

Statistics (joint-PDF)

$\ell_c = 170\eta \quad (\in \text{ ISR})$



Energy

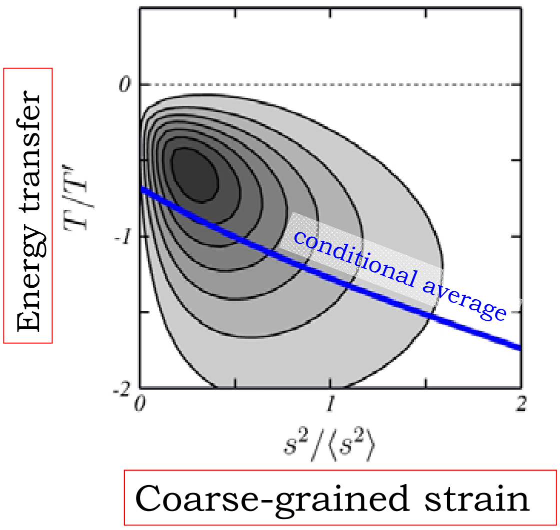
Coarse-grained enstrophy

Stronger c.-g. vorticity
→ more energy

Energy is confined in vortical regions.

Statistics (joint-PDF)

$$\ell_c = 170\eta \quad (\in \text{ ISR})$$



Stronger c.-g. strain
→ more negative energy transfer

Energy transfers in straining regions.

Another verfication

Regeneration of smaller vortices

Remove all small-scale structures
(by low-pass filtering of the Fourier modes of velocity)

↓

skip

Observe the regeneration process.

Summary of the verification of the scenario

- ✓ The scale-dependent energy is confined in the vortex tubes at the corresponding scale.

  → energy cascade is creation of smaller eddies

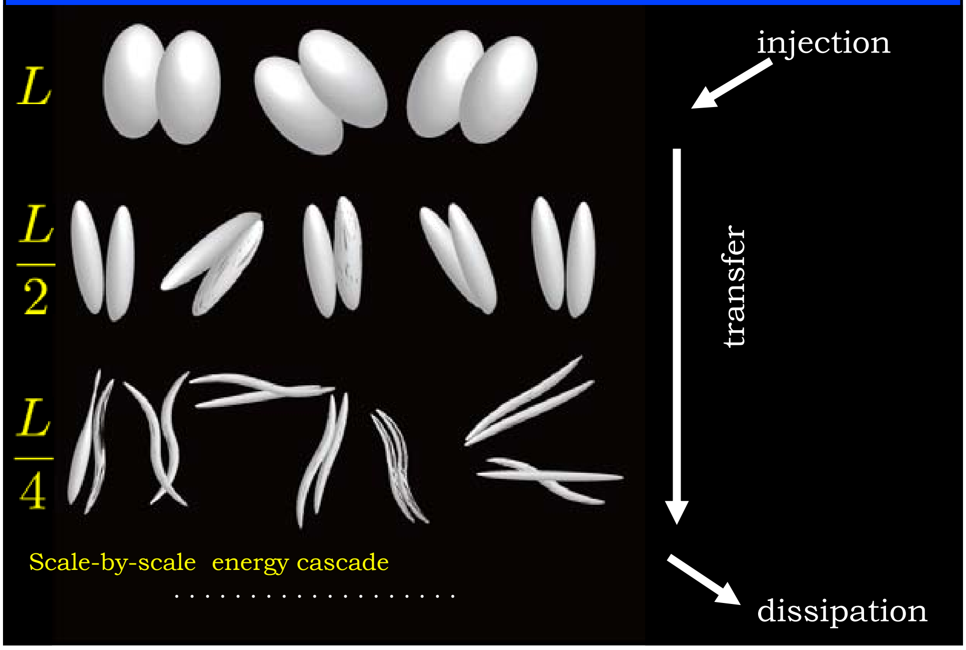
- ✓ The energy transfer takes place in the straining regions.

  → energy cascade is caused by vortex stretching

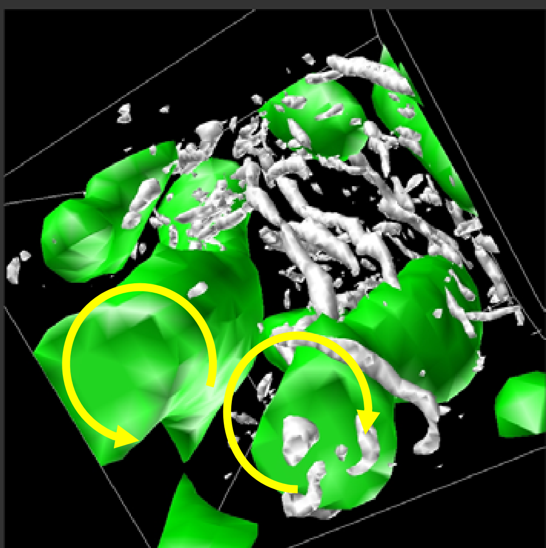
6.

Small-scale universality

Universality is due to…



Non-local energy transfer..



$$\ell_c = 340\eta$$

$$\downarrow$$

$$\ell_c = 42\eta$$

Is the cascade local in scale?

Recall that eddies are created by stretching, and diffused by viscosity.

$$\frac{D\omega}{Dt} = \omega \cdot \nabla\, u + \nu\, \nabla^2\, \omega$$

The minimum length scale of eddies created by a strain field at scale *L* can be estimated by..

the balance between the two time scales.

Is the cascade local in scale?

$$\frac{D\omega}{Dt} = \omega \cdot \nabla\, u + \nu\, \nabla^2\, \omega$$

Time scale of the strain at the integral length:

$$\tau_s = L/u'$$

Time scale of the diffusion at scale:

$$\tau_d = \ell_{\mathrm{min}}^2/\nu$$

Is the cascade local in scale?

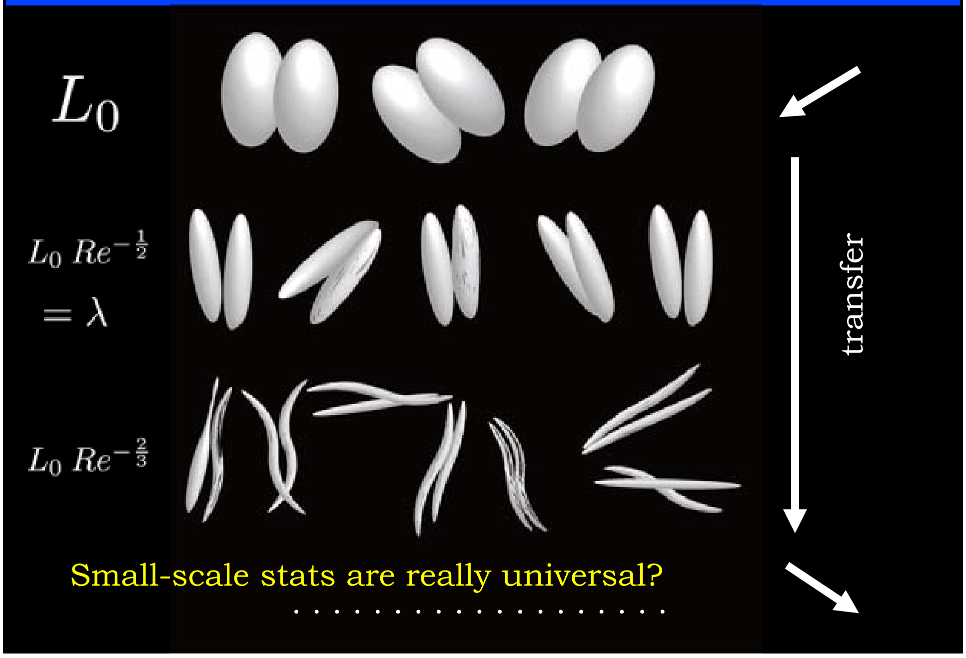
These two time scales balance:

$$\ell_{\min} = L\sqrt{\frac{\nu}{Lu'}}$$

which is so-called the Taylor "micro" scale.

Fine structures as small as the Taylor length can be created directly by the largest-scale eddies…

Non-local energy cascade

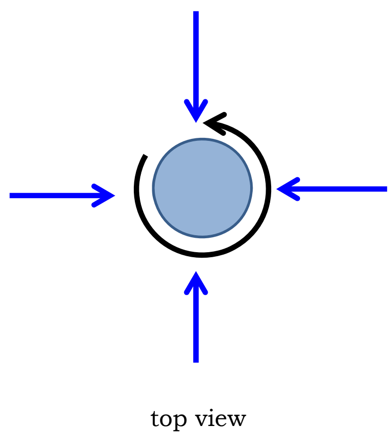


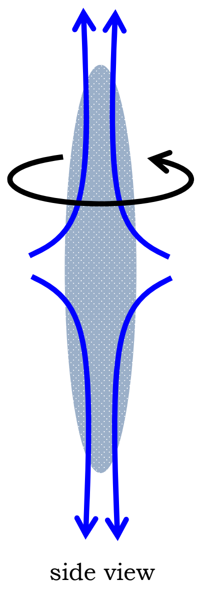
7.

Dynamics vs Statistics

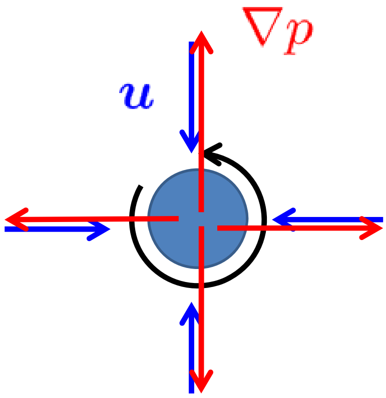
Dynamics / Statistics

If the scenario is correct…





Dynamics / Statistics



top view

Energy gain of the scale r

$$\boldsymbol{a} \cdot \boldsymbol{u} \approx -\boldsymbol{\nabla} p \cdot \boldsymbol{u}$$

$$\sim r\omega(r)^2 \times r\omega(r')$$

$$\sim r^2\omega(r)^3$$

$r' \approx r$

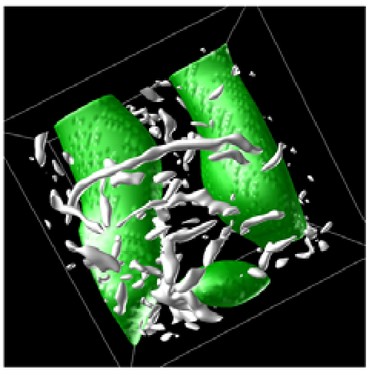
which must be ϵ

$$\omega \sim \epsilon^{\frac{1}{3}} r^{-\frac{2}{3}}$$

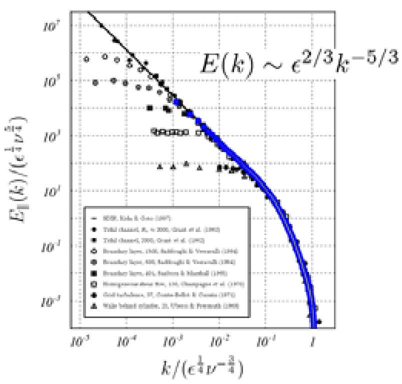
→ the -5/3 law of energy spectrum.

Dynamics / Statistics

What is the connection between
the coherent structures and the Kolmogorov law?



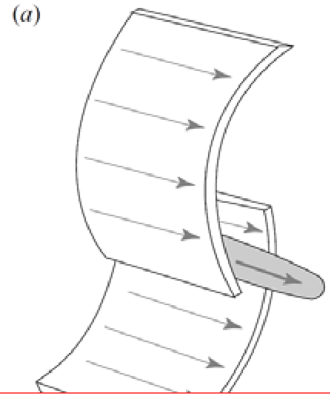
Inertial range structures.



Inertial range statistics.

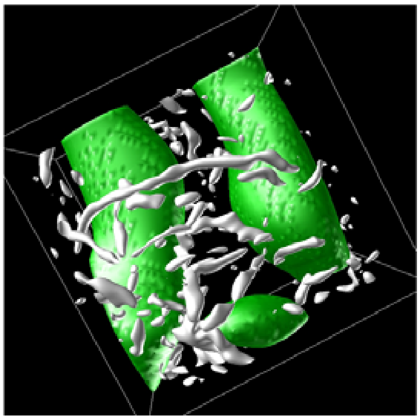
Lundgren spiral (1982)

The time averaged spectrum of the ensemble of Lundgren spirals obeys the Kolmogorov -5/3 power law.



Vorticity in spirals is **parallel** to the core vortex tube.

(taken from Horiuti & Fujisawa 2008)

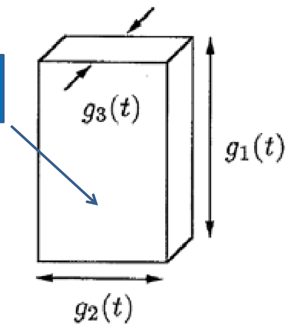


Perpendicular to the core vortex tube.

Gilbert (1993)

Generalization of the Lundgren spiral

Vortex blob



Exponential stretching

$$g_1 \sim e^{\alpha t}$$
$$g_2 \sim e^{\beta t}$$
$$g_3 \sim e^{\gamma t}$$
$$(\alpha + \beta + \gamma = 0)$$

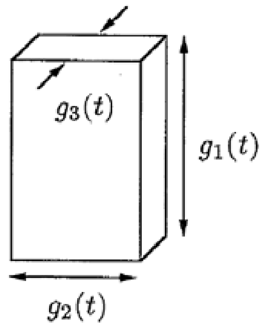
Time-averaged energy spectrum $E(k) \sim k^{-3-2\alpha/\gamma}$

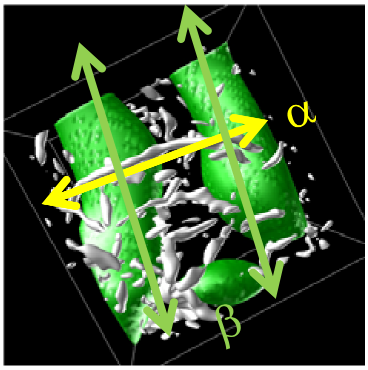
$$E(k) \sim k^{-5/3} \quad \Rightarrow \quad \alpha : \beta : \gamma = 2 : 1 : -3$$

Gilbert (1993) & CS observed in DNS

$$E(k) \sim k^{-5/3} \quad \Rightarrow \quad \alpha : \beta : \gamma = 2 : 1 : -3$$

$g_1 \sim e^{\alpha t}$
$g_2 \sim e^{\beta t}$
$g_3 \sim e^{\gamma t}$





Need further investigation..

9.

Conclusions

Conclusions

- ◆Developed turbulence consists of multi-scale coherent vortex tubes.
- ◆The cascade is the creation of thinner vortex tubes in straining fields around fatter ones.
- ◆Scale-dependent energy of fluid particles is defined to verify the scenario.
- ◆The cascade can be very non-local in scale.