Application of Granular Kinetics to Ring Processes

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Dilute, three-dimensional ring: repeat and extend Goldreich and Tremaine's calculation of the relationship between optical depth and coefficient of restitution.

Dense, two-dimensional ring: introduce and interpret a simple numerical simulation of the flow around a moonlet in the absence of gravitational interactions between the moonlet and the disk particles and between the disk particles.

Velocity distribution function: $f(\mathbf{c};\mathbf{x},t)d\mathbf{c}d\mathbf{x}$

number density: $n(\mathbf{x},t) \equiv \iiint_{\mathbf{c}} f(\mathbf{c};\mathbf{x},t)d\mathbf{c}$

Averages: $\langle \psi \rangle \equiv \frac{1}{n}\iiint_{\mathbf{c}} \psi f d\mathbf{c}$

mean velocity: $\mathbf{u} \equiv \langle \mathbf{c} \rangle = \mathbf{u}(\mathbf{x},t)$

velocity fluctuation: $\mathbf{C} \equiv \mathbf{c} - \mathbf{u} = \mathbf{C}(\mathbf{x},t)$

second moment: $\mathbf{K} \equiv \langle \mathbf{C} \otimes \mathbf{C} \rangle$

$T \equiv \mathrm{tr}(\mathbf{K})$, $\hat{\mathbf{K}} \equiv \mathbf{K} - \frac{1}{3}T\mathbf{1}$

third moment: $\mathbf{Q} \equiv \langle \mathbf{C} \otimes \mathbf{C} \otimes \mathbf{C} \rangle$

$Q_{ijk} = \frac{1}{5}\left(Q_{ipp}\delta_{jk} + Q_{jpp}\delta_{ki} + Q_{kpp}\delta_{ij}\right)$, $q_i \equiv \frac{1}{2}\rho Q_{ipp}$

Balance equations

mass: $\rho \equiv mn = \rho_s \nu, \quad \nu = \pi d^3 n / 6$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

linear momentum

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla \cdot (\rho \mathbf{K}) - \rho G M \frac{\mathbf{R}}{|\mathbf{R}|^3}$$

second moment

$$\rho \frac{\partial \mathbf{K}}{\partial t} + \rho (\mathbf{u} \cdot \nabla) \mathbf{K} + \rho (\mathbf{K} \cdot \nabla) \mathbf{u} + \rho \left[(\mathbf{K} \cdot \nabla) \mathbf{u} \right]^T + \nabla \cdot (\rho \mathbf{Q}) = \mathbf{\Gamma} [\mathbf{C} \otimes \mathbf{C}]$$

Explicit form

$$f(c; \mathbf{x}, t) = \frac{n}{(8\pi K)^{1/2}} \exp\left(-\frac{1}{2} \mathbf{C} \cdot \mathbf{K}^{-1} \mathbf{C} \right), \quad K \equiv \det(\mathbf{K})$$

Collisions

$\mathbf{c}_1$ and $\mathbf{c}_2$ pre-collisional velocities, $\mathbf{c}_1'$ and $\mathbf{c}_2'$ post-collisional velocities, unit vector $\mathbf{k}$ directed from 1 to 2, coefficient of restitution e, relative velocity $\mathbf{g} \equiv \mathbf{C}_1 - \mathbf{C}_2$, unit vector $\mathbf{j}$ in the plane of $\mathbf{g}$ and $\mathbf{k}$, perpendicular to $\mathbf{k}$.

$$\mathbf{g}' \cdot \mathbf{k} = -\mathrm{e}\left(\mathbf{g} \cdot \mathbf{k}\right)$$

$$\mathbf{c}_1' = \mathbf{c}_1 - \frac{1+\mathrm{e}}{2}\left(\mathbf{g} \cdot \mathbf{k}\right)\mathbf{k} \qquad \mathbf{c}_2' = \mathbf{c}_2 + \frac{1+\mathrm{e}}{2}\left(\mathbf{g} \cdot \mathbf{k}\right)\mathbf{k}$$

Total change of second moment

$$\begin{aligned}\boldsymbol{\Delta} &\equiv \mathbf{C}_1' \otimes \mathbf{C}_1' + \mathbf{C}_2' \otimes \mathbf{C}_2' - \mathbf{C}_1 \otimes \mathbf{C}_1 - \mathbf{C}_2 \otimes \mathbf{C}_2 \\ &= -\frac{1}{2}(1+\mathrm{e})(\mathbf{g} \cdot \mathbf{k}) \\ &\quad \times \left[(1-\mathrm{e})(\mathbf{g} \cdot \mathbf{k})\mathbf{k} \otimes \mathbf{k} + (\mathbf{g} \cdot \mathbf{j})\left(\mathbf{k} \otimes \mathbf{j} + \mathbf{j} \otimes \mathbf{k}\right)\right]\end{aligned}$$

Collisional production of second moment

$$\boldsymbol{\Gamma}[\mathbf{C}\otimes\mathbf{C}] = \frac{1}{2}\mathrm{m}\iiint_{\mathbf{g}\cdot\mathbf{k}\geq 0}\boldsymbol{\Delta}\mathrm{f}_1\mathrm{f}_2\mathrm{d}(\mathbf{g}\cdot\mathbf{k})\mathrm{d}\mathbf{k}\mathrm{d}\mathbf{c}_1\mathrm{d}\mathbf{c}_2$$

$$= -\frac{6}{\pi^{3/2}}(1+\mathrm{e})\frac{\rho\nu\mathrm{T}^{3/2}}{\mathrm{d}}\boldsymbol{\gamma}$$

$$\boldsymbol{\gamma} = (1-\mathrm{e})\mathbf{A} + 2\hat{\mathbf{B}}$$

$$\mathbf{A} \equiv \iiint_{\mathbf{g}\cdot\mathbf{k}\geq 0}\mathbf{k}\otimes\mathbf{k}(\mathbf{k}\cdot\mathbf{K}\mathbf{k}/\mathrm{T})^{3/2}\mathrm{d}\mathbf{k}$$

$$\hat{\mathbf{B}} \equiv \iiint_{\mathbf{g}\cdot\mathbf{k}\geq 0}\left(\mathbf{k}\otimes\mathbf{i}+\mathbf{i}\otimes\mathbf{k}\right)(\mathbf{k}\cdot\mathbf{K}\mathbf{k}/\mathrm{T})^{1/2}(\mathbf{k}\cdot\mathbf{K}\mathbf{i}/\mathrm{T})\mathrm{d}\mathbf{k}$$

Second moment

$$\alpha \equiv \left(K_1 - K_2\right)/2T, \qquad \beta \equiv \left[\left(K_1 + K_2\right)/2T\right] - 1$$

Cylindrical polar: $r,\ \phi,\ z$

$$\cos\chi = \mathbf{e}_r \cdot \mathbf{e}_1$$

$$\mathbf{K} = T\begin{bmatrix} 1+\beta+\alpha\cos 2\chi & \alpha\sin 2\chi & 0 \\ \alpha\sin 2\chi & 1+\beta-\alpha\cos 2\chi & 0 \\ 0 & 0 & 1-2\beta \end{bmatrix}$$

Nearly homogeneous

$\alpha,\ \beta$ and χ constant; $T = T(z),\ \nu = \nu(z),\ u \equiv u_\phi = u(r)$

Balance equations at lowest order

$$u(r) = \Omega(r) r, \qquad \Omega(r) \equiv \left(\frac{GM}{r^3} \right)^{1/2}$$

$$-4\rho\Omega K_{r\phi} = \Gamma_{rr} - \frac{\partial}{\partial z}\left(\rho Q_{zrr}\right)$$

$$\Omega K_{r\phi} = \Gamma_{\phi\phi} - \frac{\partial}{\partial z}\left(\rho Q_{z\phi\phi}\right)$$

$$0 = \Gamma_{zz} - \frac{\partial}{\partial z}\left(\rho Q_{zzz}\right)$$

$$\frac{1}{2}\rho\Omega\left(K_{rr} - 4K_{\phi\phi}\right) = \Gamma_{r\phi} - \frac{\partial}{\partial z}\left(\rho Q_{zr\phi}\right)$$

Eigenvector basis

$$-\frac{3}{2}\rho\Omega T\left(1+\beta+\alpha\right)\sin 2\chi=\Gamma_{11}-\cos^2\chi\frac{\partial}{\partial z}\left(\rho Q_{zrr}\right)$$
$$-\sin 2\chi\frac{\partial}{\partial z}\left(\rho Q_{zr\phi}\right)-\sin^2\chi\frac{\partial}{\partial z}\left(\rho Q_{z\phi\phi}\right)$$

$$\frac{3}{2}\rho\Omega T\left(1+\beta-\alpha\right)\sin 2\chi=\Gamma_{22}-\sin^2\chi\frac{\partial}{\partial z}\left(\rho Q_{zrr}\right)$$
$$+\sin 2\chi\frac{\partial}{\partial z}\left(\rho Q_{zr\phi}\right)-\cos^2\chi\frac{\partial}{\partial z}\left(\rho Q_{z\phi\phi}\right)$$

$$0=\Gamma_{33}-\frac{\partial}{\partial z}\left(\rho Q_{zzz}\right)$$

$$\frac{1}{2}\rho\Omega T\left[5\alpha-3\left(1+\beta\right)\cos 2\chi\right]=\frac{1}{2}\sin 2\chi\left[\frac{\partial}{\partial z}\left(\rho Q_{zrr}\right)\right.$$
$$\left.-\frac{\partial}{\partial z}\left(\rho Q_{z\phi\phi}\right)\right]-\cos 2\chi\frac{\partial}{\partial z}\left(\rho Q_{zr\phi}\right)$$

Integrate the last over z:

$$\cos 2\chi = \frac{5\alpha}{3\left(1+\beta\right)}$$

Integrate the isotropic part over z:

$$3\alpha\Omega\sin 2\chi \int_{-\infty}^{\infty} \nu T^{1/2} dz = \frac{6}{\pi^{3/2}}\left(1-e^2\right)\frac{\mathrm{tr}(\mathbf{A})}{d}\int_{-\infty}^{\infty} \nu^2 T^{3/2} dz$$

Using this, the 33 component is

$$(1-e)\,\mathrm{tr}(\mathbf{A}) = -\hat{\gamma}_{33}$$

With the last two, the difference between the 22 and 11 components is

$$3\hat{\gamma}_{33}\left(1+\beta\right) = \alpha\left(\gamma_{22}-\gamma_{11}\right)$$

Approximate

$$\mathrm{tr}(\mathbf{A}) \doteq \frac{2\pi}{35}\left(70 + 7\alpha^2 + 21\beta^2 - 2\alpha^2\beta + 2\beta^3 + \frac{\alpha^4}{4} + \frac{3\alpha^2\beta^2}{2} + \frac{9\beta^4}{4}\right)$$

$$\gamma_{22} - \gamma_{11} \doteq -\frac{8\pi}{35}(1+e)\alpha\left(7 + 2\beta - \frac{\alpha^2}{6} - \frac{\beta^2}{2} + \frac{2\beta^3}{11} + \frac{2\alpha^2\beta}{11}\right)$$

$$\hat{\gamma}_{33} = -\frac{4\pi}{105}(1+e)\alpha\left(42\beta + 2\alpha^2 - 6\beta^2 - 3\beta^3 - \alpha^2\beta + \frac{\alpha^4}{11} + \frac{6\alpha^2\beta^2}{11} - \frac{15\beta^4}{11}\right)$$

Solve the last two balance equations for α and β in terms of $\varepsilon \equiv 1 - e$

$$\alpha^2 = \frac{7}{2}\beta + \frac{1013}{396}\beta^2$$

$$\beta \doteq 2772\frac{8+3\varepsilon}{2960+9393\varepsilon} - 12\sqrt{11}\left[4851\frac{(8+3\varepsilon)^2}{(2960+9393\varepsilon)^2} - \frac{10\varepsilon}{2960+9393\varepsilon}\right]^{1/2}$$

Lowest order in ε: $\beta \doteq 5\varepsilon/14$ and $\alpha \doteq (5\varepsilon)^{1/2}/2$

Limitations

$$\cos 2\chi = \frac{5\alpha}{3(1+\beta)}$$

with

$$\beta \doteq \frac{5\varepsilon}{14} \text{ and } \alpha \doteq \frac{(5\varepsilon)^{1/2}}{2}$$

implies that

$$\varepsilon \le 0.3688$$

or $e \ge 0.6312 \quad (0.6270)$

Isothermal

$$0 = \frac{\partial}{\partial z}\left[(1-2\beta)\nu T\right] + \nu GM\frac{z}{r^3}$$

$$\nu = \nu_0 \exp\left[-\frac{\xi^2}{2(1-2\beta)}\right] \qquad \xi \equiv \frac{\Omega}{T^{1/2}} z$$

$$3\alpha\Omega\sin 2\chi \int_{-\infty}^{\infty} \nu dz = \frac{6}{\pi^{3/2}}\left(1-e^2\right)\frac{tr(\mathbf{A})T^{1/2}}{d}\int_{-\infty}^{\infty} \nu^2 dz$$

$$3\alpha\sin 2\chi = \frac{6}{\sqrt{2}\pi^{3/2}}\left(1-e^2\right)tr(\mathbf{A})\frac{T^{1/2}}{\Omega d}\nu_0$$

Optical depth

$$\tau \equiv \frac{3}{d}\int_0^{\infty} \nu(z)dz = 3(1-\beta)^{1/2}\left(\frac{\pi}{2}\right)^{1/2}\frac{T^{1/2}}{\Omega d}\nu_0$$

$$3\alpha\sin 2\chi = \frac{2}{\pi^2}\left(1-e^2\right)\frac{tr(\mathbf{A})}{(1-\beta)^{1/2}}\tau$$

$\nu(z)$ and $T(z)$

$$0 = \frac{\partial}{\partial z}\left(\nu T\right) + \nu z \frac{\Omega^2}{(1-2\beta)}$$

$$3\Omega\rho T\alpha \sin 2\chi = \frac{6\varepsilon(2-\varepsilon)}{\pi^{1/2} d}\rho\nu T^{3/2}\mathrm{tr}(\mathbf{A}) + 2\frac{dq_z}{dz}$$

$$q_z = \frac{5\pi^{1/2}}{4(2-\varepsilon)}\frac{\rho d}{\nu}\frac{(1-2\beta)(5-4\beta)}{d_0 - 2d_1\beta T + 4d_2\beta^2 T^2}T^{1/2}\frac{dT}{dz}$$

$$d_0 = \left(33\varepsilon - 49\right) + \frac{9}{28}(1-\varepsilon)\frac{\mathrm{tr}(\hat{\mathbf{K}}^2)}{T^2}$$

$$d_1 = \frac{4}{5}\frac{(6\varepsilon - 13)}{T} \qquad d_2 = -\frac{4}{35}\frac{(6\varepsilon - 13)}{T^2}$$

Differential Equations

$$\Theta^2 \equiv T / T_0 \qquad F \equiv \nu / \nu_0 \qquad \xi \equiv \Omega z / T^{1/2}$$

$$F' = -\frac{F}{\Theta}\left[2\Theta' + \frac{\xi}{(1-2\beta)\Theta}\right]$$

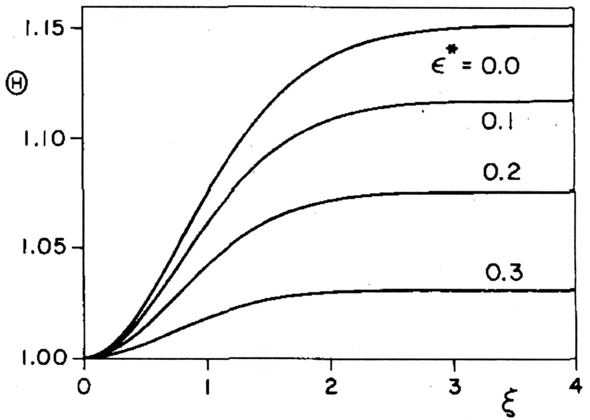
$$\left(\Theta^2\Theta'\right)' = -\frac{C_1}{S^2}F^2\Theta^3 + \frac{C_2}{S}F\Theta^2$$

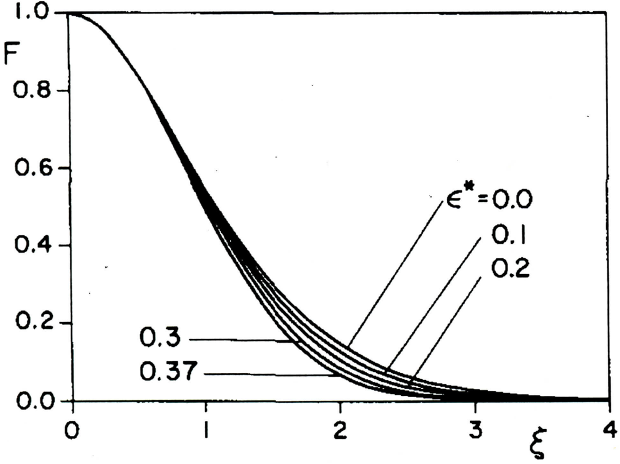
$$C_1 = C_1(\varepsilon) \qquad C_2 = C_2(\varepsilon)$$

$$S \equiv \frac{\Omega d}{\nu_0 T_0^{1/2}} = \frac{C_2\int_0^\infty F^2\Theta^3 d\xi}{C_1\int_0^\infty F\Theta^2 d\xi}$$

Initial Conditions

$$\Theta(0) = 1 \qquad \Theta'(0) = 0 \qquad F(0) = 1$$


1.15
1.10
1.05
1.00
Θ
0
1
2
3
4
ξ
ε* = 0.0
0.1
0.2
0.3



1.0
F
0.8
0.6
0.4
0.2
0.0
0
1
2
3
4
ξ
ε* = 0.0
0.1
0.2
0.3
0.37

6/7 Lindblad resonance

$$T_{6/7} = 4\pi R^2 \int_0^\infty \rho K_{rf} dz = 1.13 \times 10^{11} \text{ m}^4 \text{ s}^{-2}$$

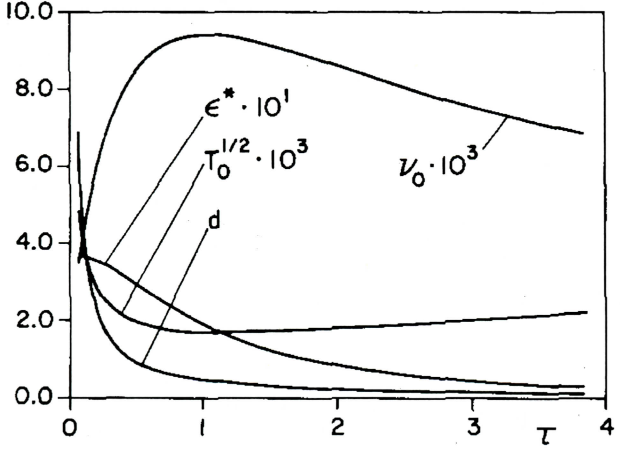
Mass density

$$\Sigma \equiv 2\int_0^\infty \rho dz = 300 \text{ kg m}^{-2}$$

$$T_0 = \frac{T_{6/7}}{2\pi \Sigma R^2} \frac{1}{\alpha \sin 2\chi} \frac{\int_0^\infty F d\xi}{\int_0^\infty F \Theta^2 d\xi}$$

$$\nu_0 = \frac{\Sigma \Omega}{2\rho_s T_0^{1/2}} \frac{1}{\int_0^\infty F d\xi}$$

$$d = \frac{\nu_0 T_0^{1/2}}{\Omega} \frac{C_2 \int_0^\infty F^2 \Theta^3 d\xi}{C_1 \int_0^\infty F \Theta^2 d\xi}$$



10.0
8.0
6.0
4.0
2.0
0.0
0
1
2
3
4
τ
ε* · 10¹
T₀^{1/2} · 10³
ν₀ · 10³
d

Shock Waves around a Moonlet in a Planar Ring

Steady, homogeneous energy balance:

$$-P_{r\phi}\dot{\gamma} + \Gamma = 0$$

$$\Gamma = \frac{4a(1-e)}{d^2}T$$

$$a \equiv \frac{2\sigma d}{\pi^{1/2}}\rho G \qquad \sigma = \frac{(1+e)}{2} \qquad G = \frac{\nu(16-7\nu)}{16(1-\nu)^2}$$

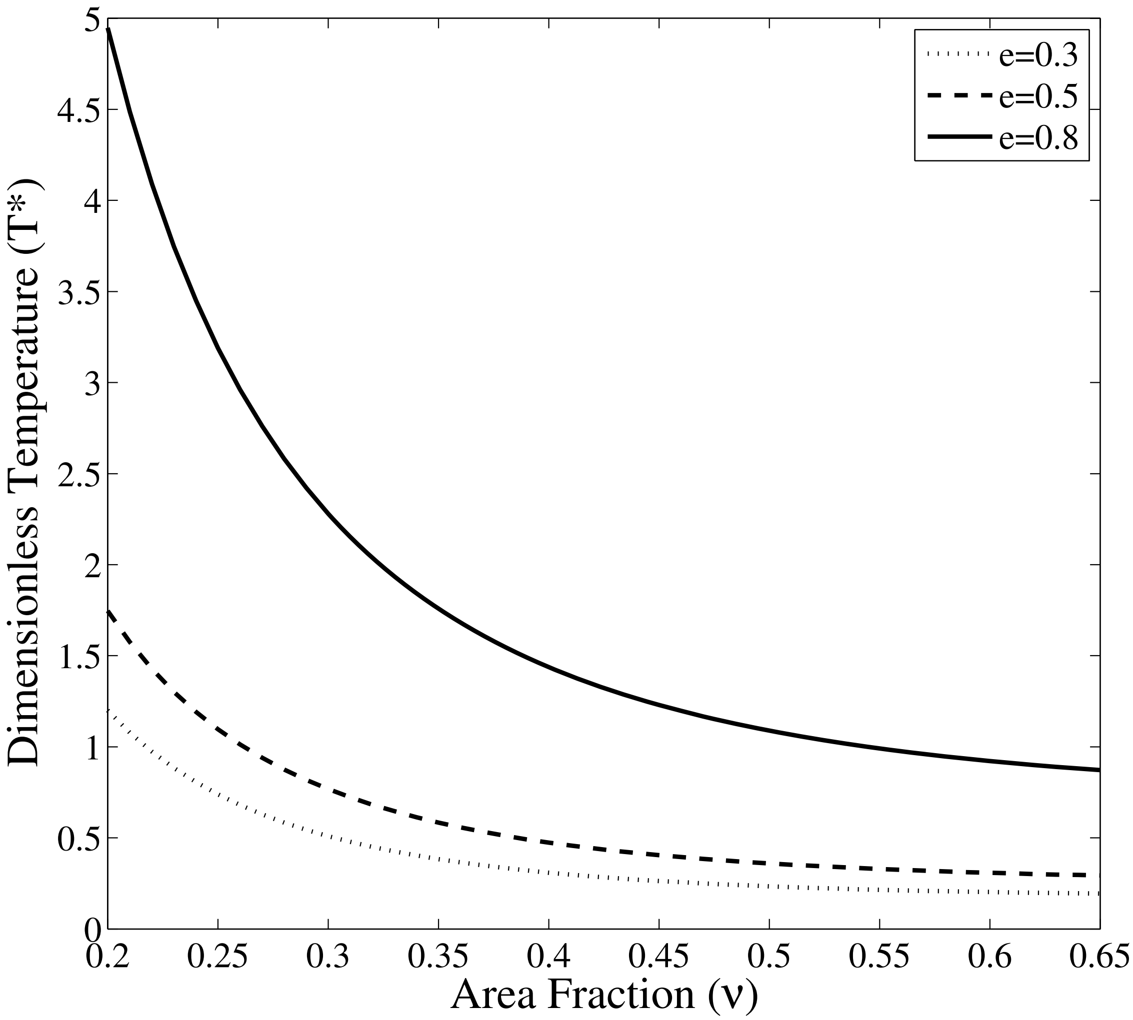
$$P_{\alpha\beta} = (p-a)\delta_{\alpha\beta} - 2\mu\hat{D}_{\alpha\beta}$$

$$p = \rho\left(1+2\sigma G\right)T$$

$$\mu = \frac{1}{2}a + \frac{\pi^{1/2}}{8}\rho d T^{1/2}\frac{\left[1-\sigma G\left(3\sigma-2\right)\right]}{(5-3\sigma)G}$$

$$\frac{T}{\dot{\gamma}^2 d^2} = \frac{1}{16(1-\sigma)}\left[1+\frac{\pi}{4}\frac{1-G\sigma+G(3-2G)\sigma^2+3G^2\sigma^3}{G^2\sigma(5-3\sigma)}\right]$$

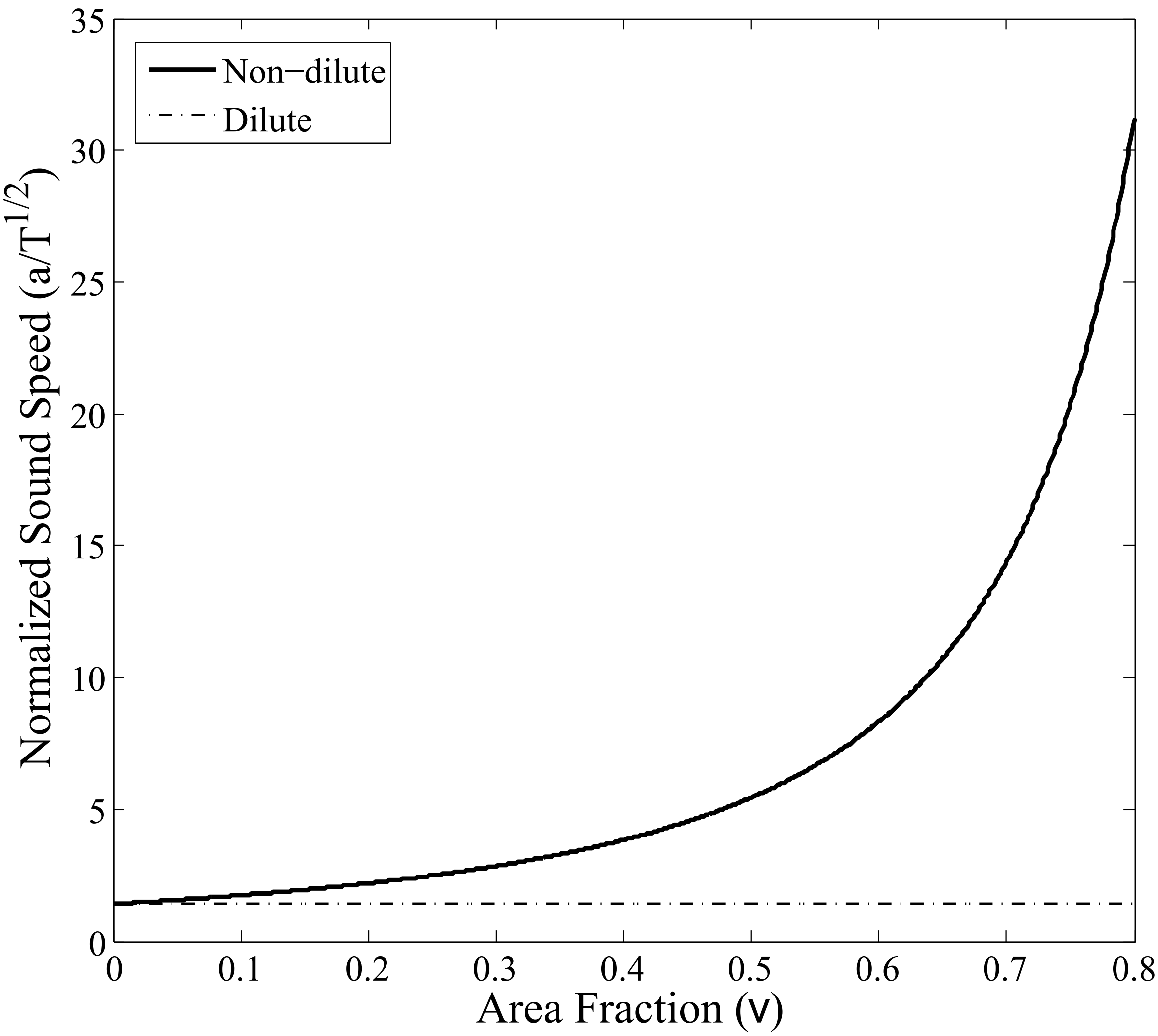
Dimensionless ring temperature $T^* \equiv T/(\dot{\gamma}d)$ versus area fraction ν



Isentropic Sound Speed

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{9\nu^4 - 32\nu^3 - 24\nu^2 + 128}{64(1-\nu)^4} T$$

$\nu = 0.2,\ 0.8$



Mach Number

$$u = \dot{\gamma} y$$

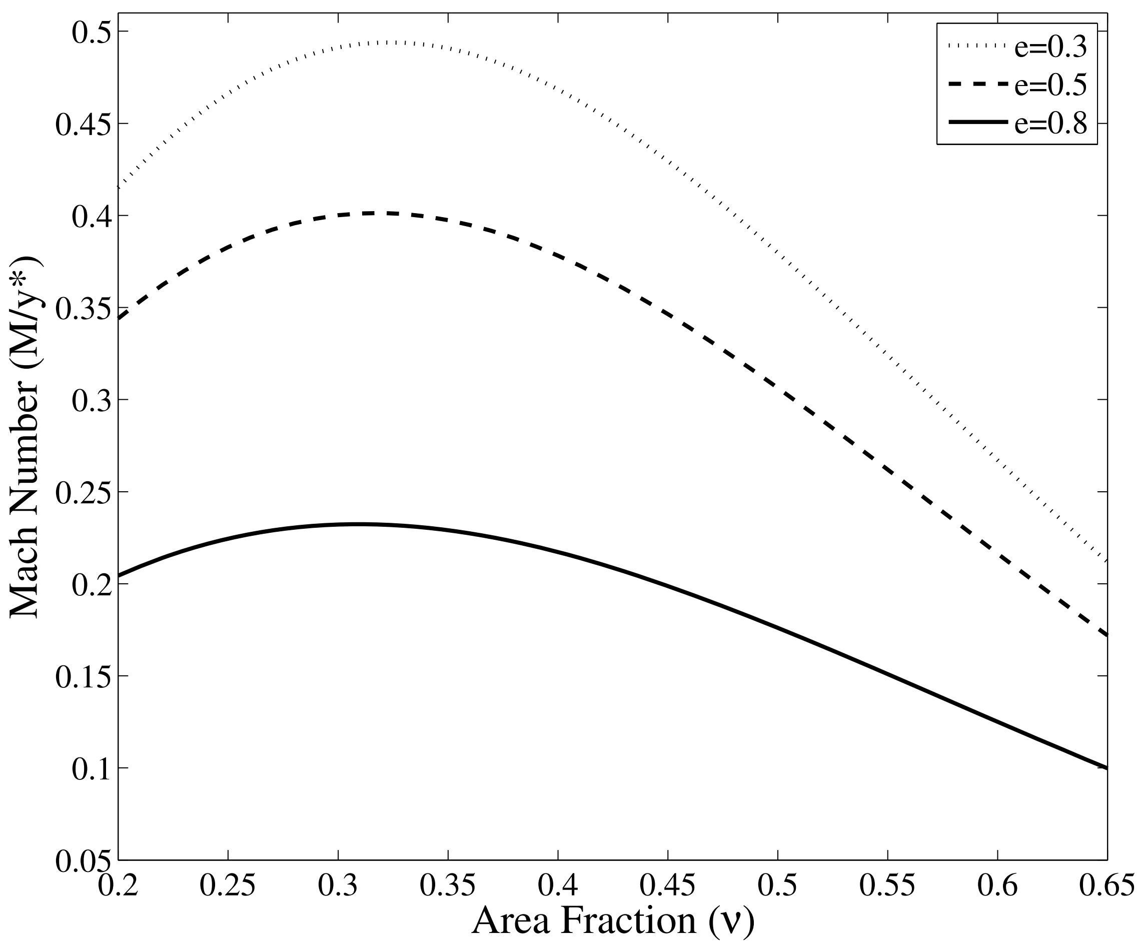
$$\frac{u^2}{T} = \frac{64G^2\left(5\sigma - 8\sigma^2 + 3\sigma^3\right)}{\pi + G\left[(20G - \pi)\sigma + (3\pi - 2\pi G - 12G)\sigma^2 + 3\pi G\sigma^3\right]}\left(\frac{y}{d}\right)^2$$

$$\frac{a^2}{T} = \frac{9\nu^4 - 32\nu^3 - 24\nu^2 + 128}{64(1-\nu)^4}$$

Mach Number M

$$M^2 = \frac{u^2 / T}{a^2 / T}$$

Mach number $M \equiv u / a$, with $a^2 \equiv \left(\partial p / \partial \rho\right)_s$, normalized by dimensionless vertical displacement, $y^* \equiv y / d$, versus sound speed



Simulations

Two-dimensional flow of identical, frictionless, circular disks

Event-driven simulations of hard-particles

Homogeneous Hill equations

$$\frac{d^2y}{dt^2} + 2\Omega\frac{dx}{dt} - 3\Omega^2 y = 0$$

$$\frac{d^2x}{dt^2} - 2\Omega\frac{dy}{dt} = 0$$

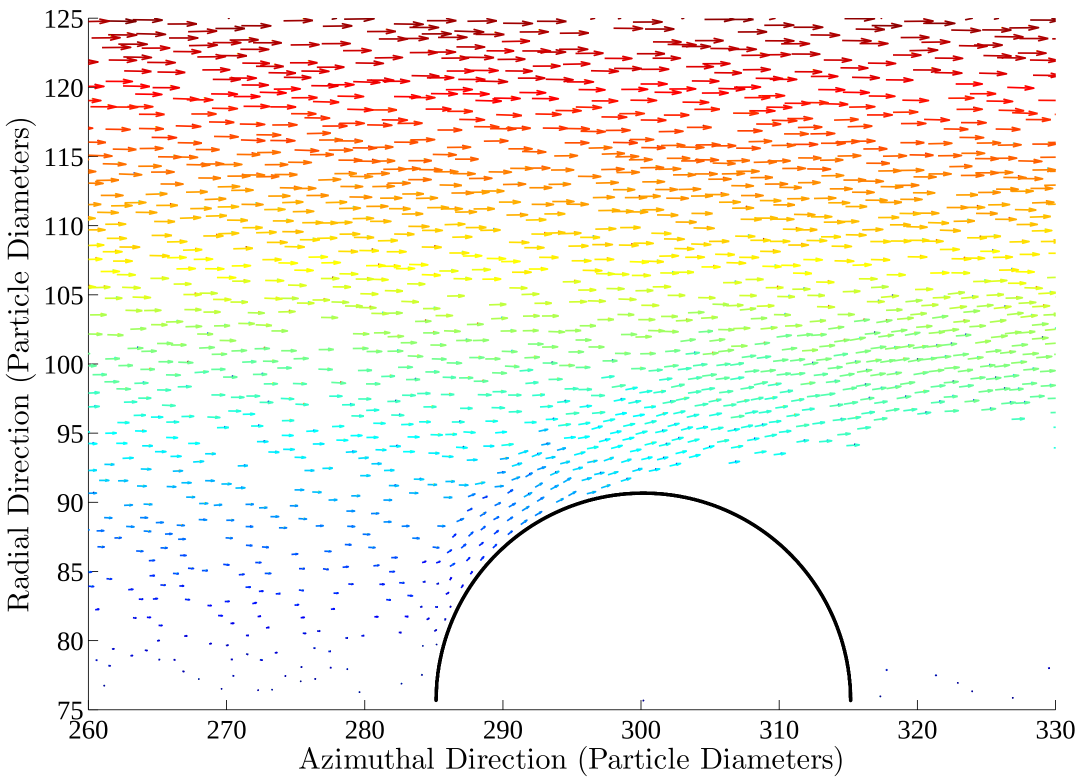
Time made dimensionless by Ω, lengths by d

moonlet diameter D

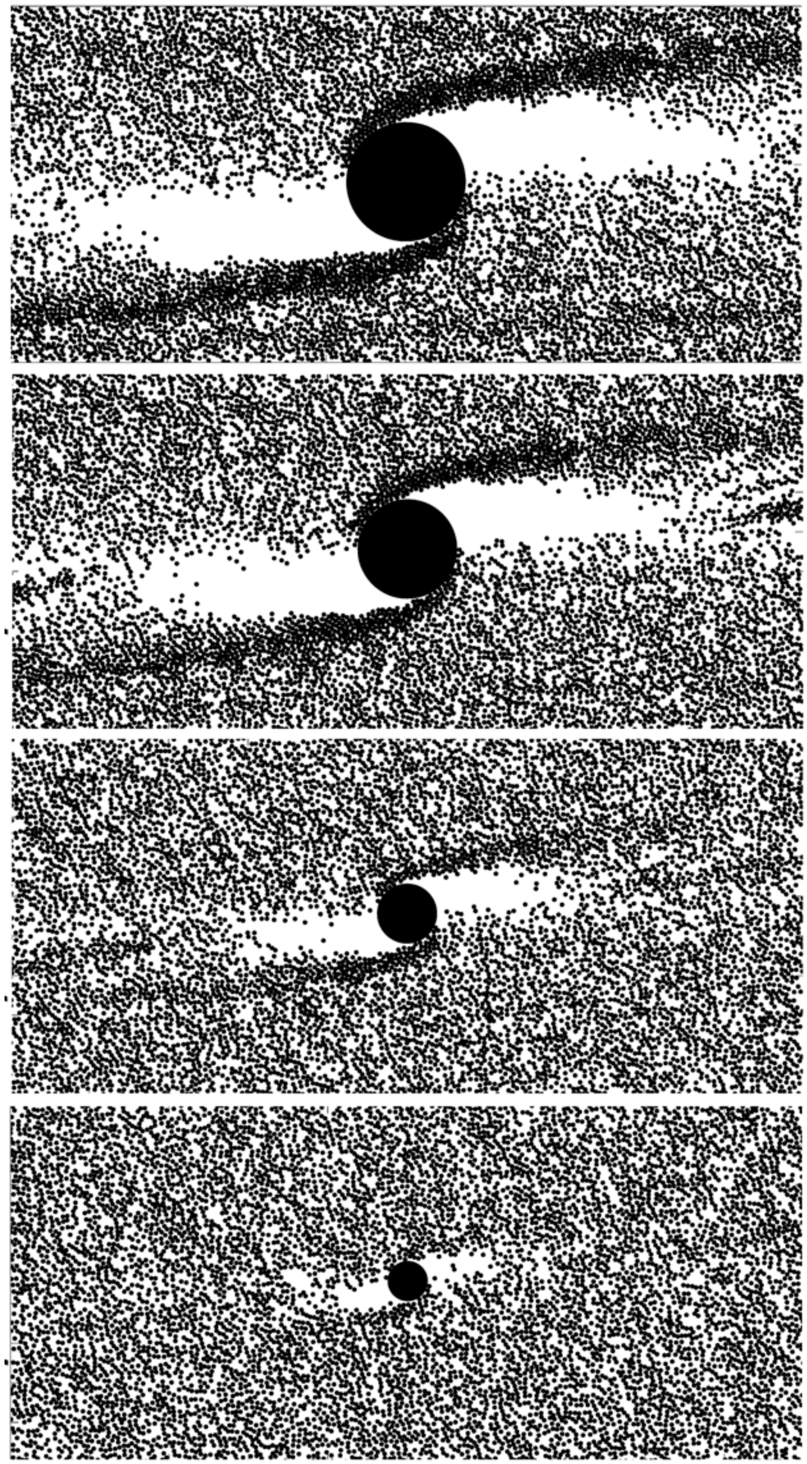
Parameters: D/d, e, ν

Particle velocity

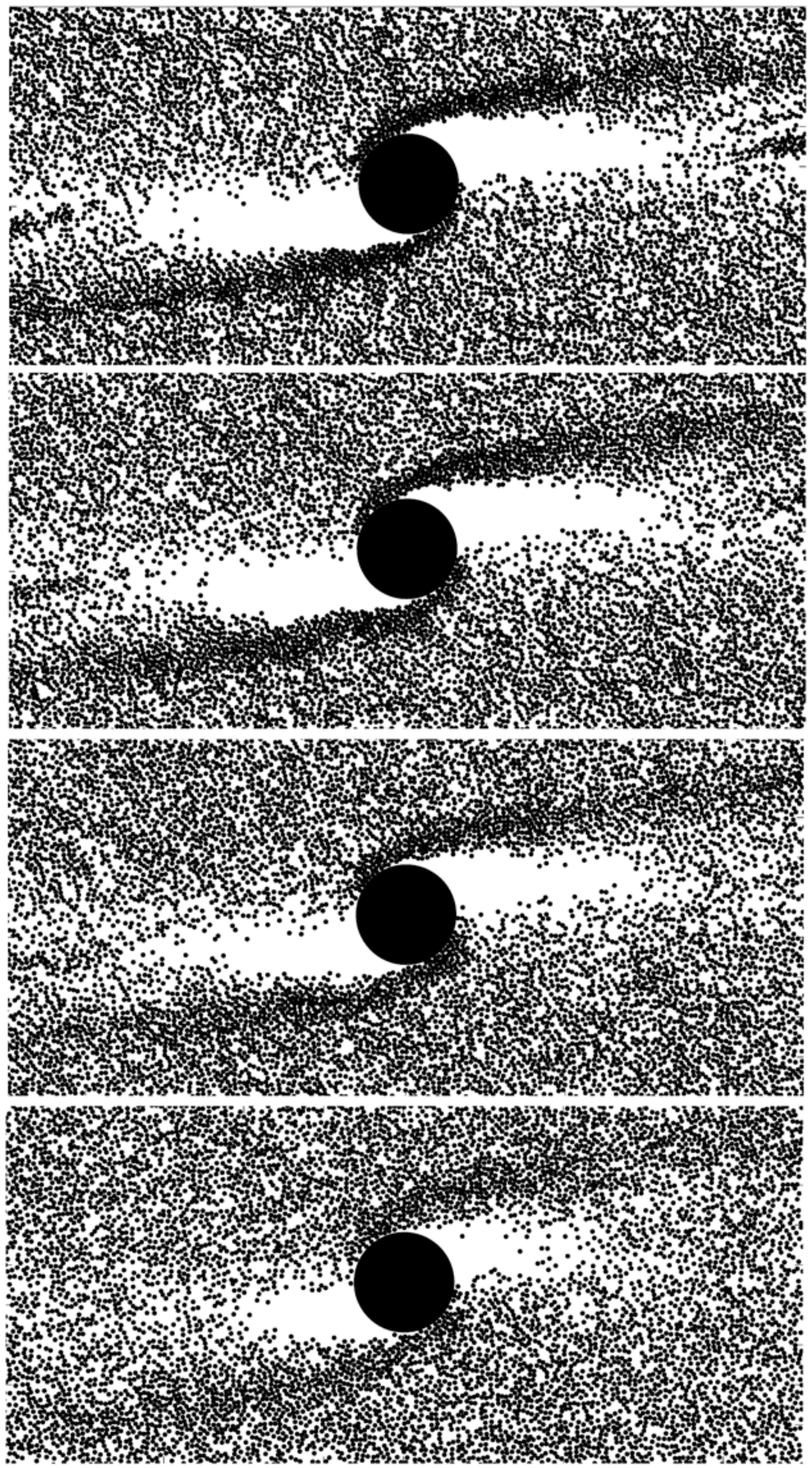
D/d = 25, ν = 0.5, e = 0.3



$D / d = 30,\ 25,\ 15,\ 10$ with $e = 0.3$ and $\nu = 0.5$



$e = 0.3,\ 0.5,\ 0.6,\ 0.8$ with $D/d = 25$ and $\nu = 0.5$



$\nu = 0.7,\ 0.5,\ 0.3$ with $e = 0.3$ and $D/d = 25$