New forms of non-relativistic and relativistic hydrodynamic equations as derived by the RG method

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- RG derivation of the rel. 1st-order diss. Hydrodynamic equations
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1.K. Tsumura, K. Ohnishi and T.K., Phys. Lett. B646 (2007), 134,
2.K. Tsumura and T.K. , arXiv:1108.1519 [hep-ph],
to be published in PTP.
3. K. Tsumura and T. K. in preparation

Introduction

- Relativistic hydrodynamics for a perfect fluid is widely and successfully used in the RHIC phenomenology. T. Hirano, D.Teaney, …

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- A growing interest in dissipative hydrodynamics.

hadron corona (rarefied states); Hirano et al …

Generically, an analysis using dissipative hydrodynamics is needed even to show the dissipative effects are small.

A.Muronga and D. Rischke; A. K. Chaudhuri and U. Heinz,; R. Baier,
P. Romatschke and U. A. Wiedemann; R. Baier and P. Romatschke (2007)
and the references cited in the last paper.

However,

is the theory of relativistic hydrodynamics for a viscous fluid fully established?

The answer is

No!

unfortunately.

Cf. T. Hirano's talk

Fundamental problems with relativistic hydro-dynamical equations for viscous fluid

a. Ambiguities in the form of the equation, even in the same frame and equally derived from Boltzmann equation: Landau frame; unique,
Eckart frame; Eckart eq. v.s. Grad-Marle-Stewart eq.; Muronga v.s. R. Baier et al

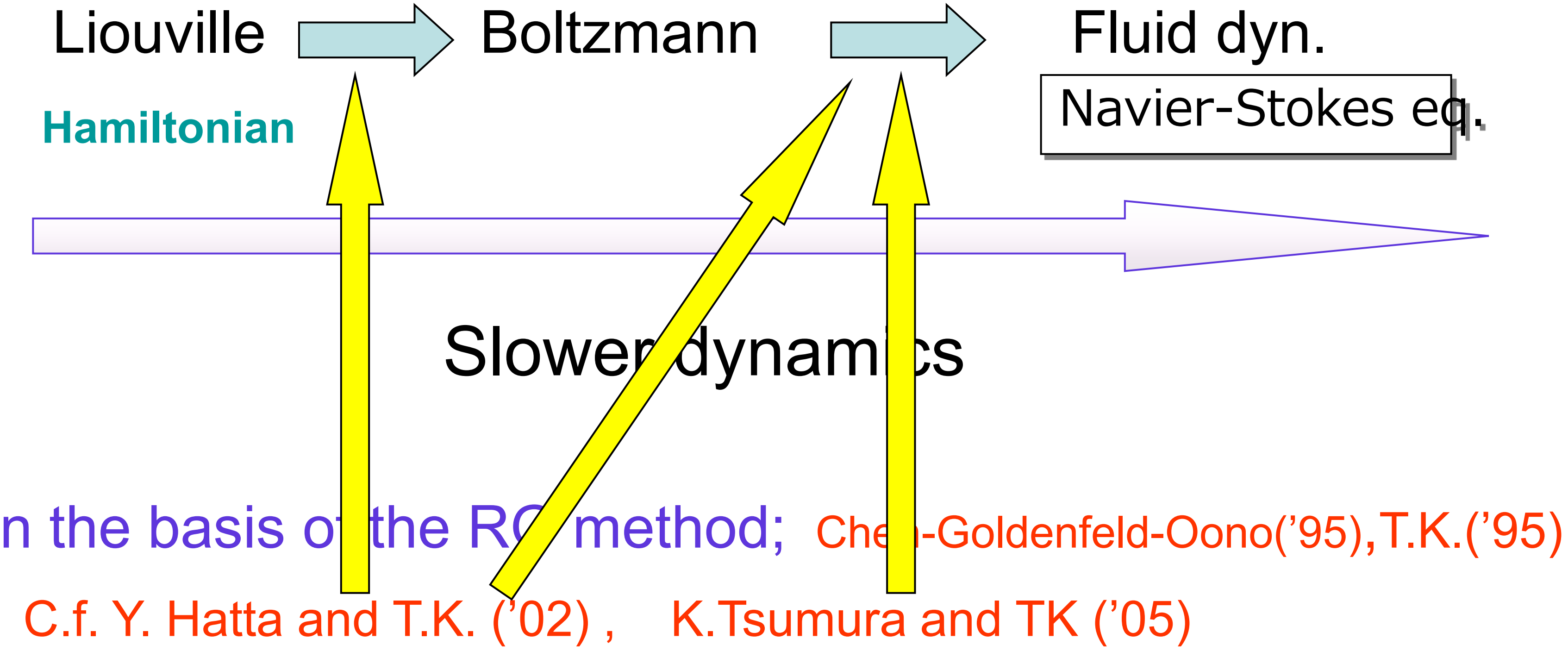
b. Instability of the equilibrium state in the eq.'s in the Eckart frame, which affects even the solutions of the causal equations, say, by Israel-Stewart.
W. A. Hiscock and L. Lindblom ('85, '87); R. Baier et al ('06, '07)

c. Usual 1st-order equations are acausal as the diffusion eq. is, except for Israel-Stewart and those based on the extended thermodynamics with relaxation times, but the form of causal equations is still controversial.

---- The purpose of the present talk ---

For analyzing the problems a and b first,
we derive hydrodynaical equations for a viscous fluid from Boltzmann equation
on the basis of a mechanical reduction theory (so called the RG method) and a natural ansatz on the origin of dissipation.
We also show that the new equation in the Eckart frame is stable.
We then proceeds to the causality problem..

The separation of scales
in the relativistic heavy-ion collisions



Liouville → Boltzmann → Fluid dyn.

Hamiltonian

Navier-Stokes eq.

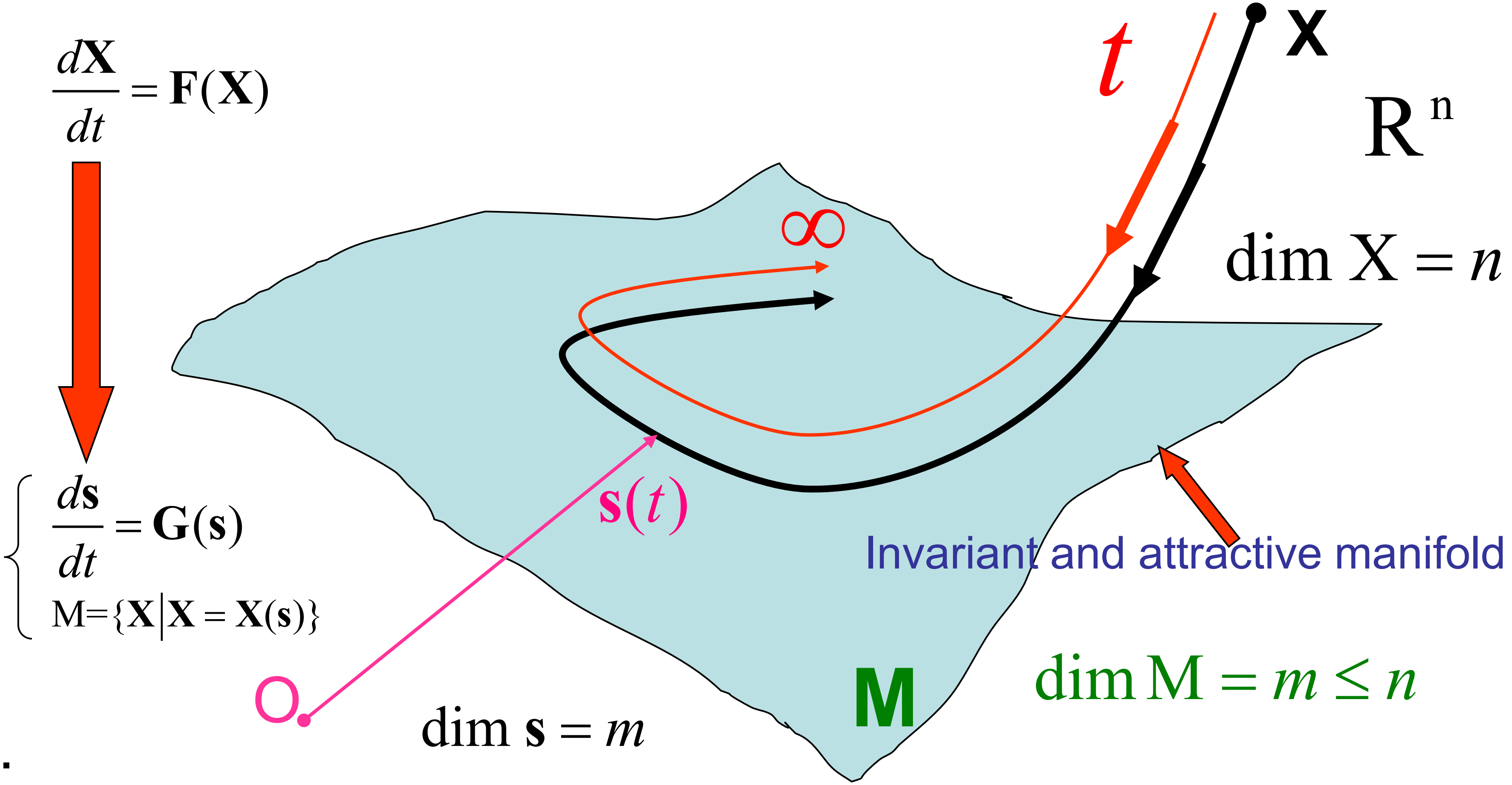
Slower dynamics

on the basis of the RG method; Chen-Goldenfeld-Oono('95),T.K.('95)

C.f. Y. Hatta and T.K. ('02) , K.Tsumura and TK ('05)

Hydrodynamics is the effective dynamics of the kinetic (Boltzmann) equation in the infrared refime.

Geometrical image of reduction of dynamics



$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X})$$

$$\begin{cases} \dfrac{d\mathbf{s}}{dt} = \mathbf{G}(\mathbf{s}) \\ \mathrm{M}=\{\mathbf{X} \,|\, \mathbf{X} = \mathbf{X}(\mathbf{s})\} \end{cases}$$

t

X

R^{n}

∞

$\dim \mathrm{X} = n$

$\mathbf{s}(t)$

Invariant and attractive manifold

O

$\dim \mathbf{s} = m$

M

$\dim \mathrm{M} = m \leq n$

eg.

$\mathbf{X} = f(\mathbf{r},\mathbf{p})$; **distribution function in the phase space (infinite dimensions)**

$\mathbf{s} = \{u^{\mu}, T, n\}$; **the hydrodinamic quantities (5 dimensions), conserved quantities.**

Relativistic Boltzmann equation

$$p^\mu \, \partial_\mu \, f_p(x) = C[f]_p(x),$$

Collision integral: $C[f]_p(x) \equiv \frac{1}{2!} \sum_{p_1} \frac{1}{p_1^0} \sum_{p_2} \frac{1}{p_2^0} \sum_{p_3} \frac{1}{p_3^0} \, \omega(p\,,\, p_1 | p_2\,,\, p_3) \Big(f_{p_2}(x)\, f_{p_3}(x) - f_p(x)\, f_{p_1}(x) \Big),$

Symm. property of the transition probability:

$$\omega(p\,,p_1|p_2\,,p_3) = \omega(p_2\,,p_3|p\,,p_1) = \omega(p_1\,,p|p_3\,,p_2) = \omega(p_3\,,p_2|p_1\,,p) \qquad \text{--- (1)}$$

Energy-mom. conservation; $\omega(p\,,\, p_1 | p_2\,,\, p_3) \propto \delta^4(p + p_1 - p_2 - p_3)$ --- (2)

Owing to (1),

$$\sum_p \frac{1}{p^0} \varphi_p(x)\, C[f]_p(x) = \frac{1}{2!} \sum_p \frac{1}{p^0} \sum_{p_1} \frac{1}{p_1^0} \sum_{p_2} \frac{1}{p_2^0} \sum_{p_3} \frac{1}{p_3^0} \frac{1}{4} \Big[\omega(p\,,\, p_1 | p_2\,,\, p_3) \Big(\varphi_p(x) + \varphi_{p_1}(x) - \varphi_{p_2}(x) - \varphi_{p_3}(x) \Big) \times \Big(f_{p_2}(x)\, f_{p_3}(x) - f_p(x)\, f_{p_1}(x) \Big) \Big]. \qquad (3)$$

Collision Invariant $\varphi_p(x)$ **:** $\sum_p \frac{1}{p^0} \varphi_p(x)\, C[f]_p(x) = 0,$

Eq.'s (3) and (2) tell us that

the general form of a collision invariant; $\varphi_p(x) = \alpha(x) + p^\mu \, \beta_\mu(x),$

which can be x-dependent!

Local equilibrium distribution

The entropy current:
$$S^\mu(x) \equiv -\sum_p \frac{1}{p^0} p^\mu\, f_p(x)\, (\ln f_p(x) - 1)$$
$$\partial_\mu S^\mu(x) = -\sum_p \frac{1}{p^0} C[f]_p(x)\, \ln f_p(x).$$

Conservation of entropy $\longrightarrow$ $\ln f_p(x) = \alpha(x) + p^\mu\, \beta_\mu(x),$

$$f_p(x) = \frac{1}{(2\pi)^3} \exp\left[\frac{\mu(x) - p^\mu\, u_\mu(x)}{T(x)}\right] \equiv f_p^{\rm eq}(x)$$

i.e., **the local equilibrium distribution fn;**

(Maxwell-Juettner dist. fn.)

Remark:

Owing to the energy-momentum conservation,
the collision integral also vanishes for the local equilibrium distribution fn.;

$$C[f_p{}^{eq}](x) = 0.$$

The standard method

---Use of conditions of fit ---

$$\delta n = u_\mu \left[\sum_p \frac{1}{p^0} p^\mu \, \delta f_p \right] = 0,$$

$$\delta e = u_\mu \left[\sum_p \frac{1}{p^0} p^\mu \, p^\nu \, \delta f_p \right] u_\nu = 0.$$

$$f_p(x) = f_p^{(0)}(x) + \delta f_p(x)$$

Moreover,

For, particle frame
$$\nu^\mu = \Delta^{\mu\nu}_{\mathrm{CE}} \, \delta N_\nu = \Delta^{\mu\nu}_{\mathrm{CE}} \left[\sum_p \frac{1}{p^0} p_\nu \, \delta \, f_p \right] = 0$$

For, energy frame
$$Q^\mu = \Delta^{\mu\nu}_{\mathrm{CE}} \, \delta T_{\nu\rho} \, u^\rho = \Delta^{\mu\nu}_{\mathrm{CE}} \left[\sum_p \frac{1}{p^0} p_\nu \, p_\rho \, \delta \, f_p \right] u^\rho = 0$$

Previous attempts to derive the dissipative hydrodynamics as a reduction of the dynamics

N.G. van Kampen, J. Stat. Phys. 46(1987), 709

unique but non-covariant form and hence not Landau either Eckart!

Cf. Chapman-Enskog method to derive Landau and Eckart eq.'s; see, eg, de Groot et al ('80)

Here,

In the covariant formalism,
in a unified way and systematically
derive dissipative rel. hydrodynamics at once!

Derivation of the relativistic hydrodynamic equation from the rel. Boltzmann eq. --- an RG-reduction of the dynamics

K. Tsumura, T.K. K. Ohnishi; Phys. Lett. B646 (2007) 134-140

c.f. Non-rel. Y.Hatta and T.K., Ann. Phys. 298 ('02), 24; T.K. and K. Tsumura, J.Phys. A:39 (2006), 8089

Ansatz of the origin of the dissipation= the spatial inhomogeneity,

leading to Navier-Stokes in the non-rel. case .

$\boldsymbol{a}_p^\mu$ would become a macro flow-velocity ⟵ **Coarse graining of space-time**

$\boldsymbol{a}_p^\mu$ may not be u^μ

$$\tau \equiv \boldsymbol{a}_p^\mu x_\mu, \quad \sigma^\mu \equiv \left(g^{\mu\nu} - \frac{\boldsymbol{a}_p^\mu \boldsymbol{a}_p^\nu}{\boldsymbol{a}_p^2} \right) x_\nu \equiv \boldsymbol{\Delta}_p^{\mu\nu} x_\nu \qquad x^\mu \Longrightarrow \tau \quad \sigma^\mu$$

$$\frac{\partial}{\partial \tau} = \frac{1}{\boldsymbol{a}_p^2} \boldsymbol{a}_p^\mu \partial_\mu \equiv \boldsymbol{D},$$ time-like derivative

$$\boldsymbol{\Delta}_p^{\mu\nu} \frac{\partial}{\partial \sigma^\nu} = \boldsymbol{\Delta}_p^{\mu\nu} \partial_\nu \equiv \boldsymbol{\nabla}^\mu$$ space-like derivative

Rewrite the Boltzmann equation as,

$$\Longrightarrow \quad \frac{\partial}{\partial \tau} f_p(\tau\,,\,\sigma) = \frac{1}{p\cdot \boldsymbol{a}_p} C[f]_p(\tau\,,\,\sigma) - \frac{1}{p\cdot \boldsymbol{a}_p} p \cdot \boldsymbol{\nabla} f_p(\tau\,,\,\sigma)$$

perturbation

Only spatial inhomogeneity leads to dissipation.

RG gives a resummed distribution function, from which $T^{\mu\nu}$ and N^μ are obtained.

Chen-Goldenfeld-Oono('95), T.K.('95), **S.-I. Ei, K. Fujii and T.K. (2000)**

Examples

- $\boxed{\theta = 0}$

$$\Longleftrightarrow \quad \boldsymbol{a}_p^\mu = u^\mu$$

$$\partial_\mu J^{\mu\alpha}_{\text{hydro.}} = 0 \qquad \boxed{p \equiv nT}$$

$$\Delta J^{\mu\alpha} = \begin{cases} -\zeta\,\Delta^{\mu\nu} X + 2\,\eta\,X^{\mu\nu} & \alpha = \nu \\ -T\,\lambda\,z\,\hat{h}^{-1} X^\mu & \alpha = 4. \end{cases}$$

→ satisfies the Landau constraints

$$u_\mu u_\nu \delta T^{\mu\nu} = 0, u_\mu \Delta_{\sigma\nu} \delta T^{\mu\nu} = 0$$

$$u_\mu \delta N^\mu = 0$$

$$\begin{aligned} X &\equiv -\nabla_\mu u^\mu, \\ X_\mu &\equiv \nabla_\mu \ln T - \hat{h}^{-1} \nabla_\mu \ln(n\,T), \\ X_{\mu\nu} &\equiv \frac{1}{2}\left(\Delta_{\mu\rho}\,\Delta_{\nu\sigma} + \Delta_{\mu\sigma}\,\Delta_{\nu\rho} - \frac{2}{3}\Delta_{\mu\nu}\,\Delta_{\rho\sigma}\right)\nabla^\rho u^\sigma. \end{aligned}$$

$$T^{\mu\nu} = \epsilon\,u^\mu\,u^\nu - (p + \zeta\,X)\,\Delta^{\mu\nu} + 2\,\eta\,X^{\mu\nu}$$

$$N^\mu = n\,u^\mu - \lambda\,\frac{n\,T}{\epsilon + p}\,X^\mu.$$

Landau frame and Landau eq.!

with the microscopic expressions for the transport coefficients;

Bulk viscosity $$\zeta \;\equiv\; -\frac{1}{T}\sum_{pq}\frac{1}{p^0}\,f_p^{\mathrm{eq}}\,\Pi_p\,\mathcal{L}^{-1}_{pq}\,\Pi_q,$$

Heat conductivity $$\lambda \;\equiv\; -\frac{1}{3}\,\frac{1}{T^2}\sum_{pq}\frac{1}{p^0}\,f_p^{\mathrm{eq}}\,Q_p^{\mu}\,\mathcal{L}^{-1}_{pq}\,Q_{\mu q},$$

Shear viscosity $$\eta \;\equiv\; -\frac{1}{10}\,\frac{1}{T}\sum_{pq}\frac{1}{p^0}\,f_p^{\mathrm{eq}}\,\Pi_p^{\mu\nu}\,\mathcal{L}^{-1}_{pq}\,\Pi_{\mu\nu q}$$

$$\mathcal{L}_{pq} \;\equiv\; (p\cdot\theta_p)\,L_{pq} \;\Longleftarrow\; \theta_p \text{ -independent}$$

c.f. $$L_{pq} = -\frac{1}{p\cdot a_p}\,\frac{1}{2!}\sum_{p_1}\frac{1}{p_1^0}\sum_{p_2}\frac{1}{p_2^0}\sum_{p_3}\frac{1}{p_3^0}\,\omega(p\,,\,p_1|p_2\,,\,p_3)\,f_{p_1}^{\mathrm{eq}}\left(\delta_{pq}+\delta_{p_1q}-\delta_{p_2q}-\delta_{p_3q}\right)$$

$$(a_p{}^{\mu} = \theta_p{}^{\mu})$$

In a Kubo-type form;

$$\zeta \;\equiv\; \frac{1}{T}\int_0^{\infty}\mathrm{d}s\,\langle\,\Pi(0)\,,\,\Pi(s)\,\rangle_{\mathrm{eq}},$$

$$\lambda \;\equiv\; -\frac{1}{3}\,\frac{1}{T^2}\int_0^{\infty}\mathrm{d}s\,\langle\,Q^{\mu}(0)\,,\,Q_{\mu}(s)\,\rangle_{\mathrm{eq}},$$

$$\eta \;\equiv\; \frac{1}{10}\,\frac{1}{T}\int_0^{\infty}\mathrm{d}s\,\langle\,\Pi^{\mu\nu}(0)\,,\,\Pi_{\mu\nu}(s)\,\rangle_{\mathrm{eq}}.$$

$$\left[\Pi(s)\right]_p \equiv \textstyle\sum_q \left[\mathrm{e}^{s\,\mathcal{L}}\right]_{pq}\,\Pi_q$$

$$\langle\,\varphi\,,\,\psi\,\rangle_{\mathrm{eq}} \equiv \sum_p \frac{1}{p^0}\,f_p^{\mathrm{eq}}\,\varphi_p\,\psi_p$$

C.f. Bulk viscosity may play a role in determining the acceleration of the expansion of the universe, and hence the dark energy!

Eckart (particle-flow) frame:

Landau equation: $a_p^\mu = u^\mu$

Setting $a_p^\mu = \frac{m}{p\cdot u} u^\mu$

$$T^{\mu\nu} = (\epsilon + 3\zeta\tilde{X})u^\mu u^\nu - (p + \zeta\tilde{X})\Delta^{\mu\nu} + \lambda T u^\mu \tilde{X}^\nu + \lambda T u^\nu \tilde{X}^\mu + 2\eta X^{\mu\nu}$$

$$N^\mu = m n u^\mu$$

i.e., $\delta N^\mu = 0.$

with

$$\tilde{X} \equiv -\{1/3\,(4/3 - \gamma)^{-1}\}^2 \nabla\cdot u$$

$$\tilde{X}^\mu \equiv \nabla^\mu \ln T.$$

(i) This satisfies the GMS constraints but not the Eckart's.

(ii) Notice that only the space-like derivative is incorporated.

(iii) This form is different from Eckart's and Grad-Marle-Stewart's, both of which involve the time-like derivative.

Eckart's constraints :

1. $u_\mu u_\nu \delta T^{\mu\nu} = 0,$
2. $u_\mu \delta N^\mu = 0,$
3. $\Delta_{\mu\nu}\delta N^\nu = 0,$

⟺

5. $T^\mu_{\ \mu} = 0,$
2. $u_\mu \delta N^\mu = 0,$
3. $\Delta_{\mu\nu}\delta N^\nu = 0.$

Grad-Marle-Stewart constraints

c.f. Grad-Marle-Stewart equation;

$$\delta T^{\mu\nu} = -3\,(3T^{-1}C_T + 1)^{-1}\zeta u^\mu u^\nu \nabla\cdot u + u^\mu T\lambda\left(\frac{1}{T}\nabla^\nu T - Du^\nu\right) + u^\nu T\lambda\left(\frac{1}{T}\nabla^\mu T - Du^\mu\right)$$
$$+ 2\eta\frac{1}{2}\left(\nabla^\mu u^\nu + \nabla^\nu u^\mu - \frac{2}{3}\Delta^{\mu\nu}\nabla\cdot u\right) + (3T^{-1}C_T + 1)^{-1}\zeta\Delta^{\mu\nu}\nabla\cdot u,$$

$$\delta N^\mu = 0.$$

The stability of the solutions in the particle frame:

K.Tsumura and T.K. (2008)

(i) The Eckart and Grad-Marle-Stewart equations gives an instability, which has been known, and is now found to be attributed to the fluctuation-induced dissipation, proportional to Du^μ.

(ii) Our equation (TKO equation) seems to be stable, being dependent on the values of the transport coefficients and the EOS.

The numerical analysis shows that, the solution to our equation is stable at least for rarefied gasses.

A comment:
our equations derived by the RG method naturally ensure the stability of the thermal equilibrium state;
this is a consequence of the positive-definiteness of the inner product.
(K. Tsumura and T.K., (2011)), PTP, to be published.

II Second-order equations and moment method

Purpose:

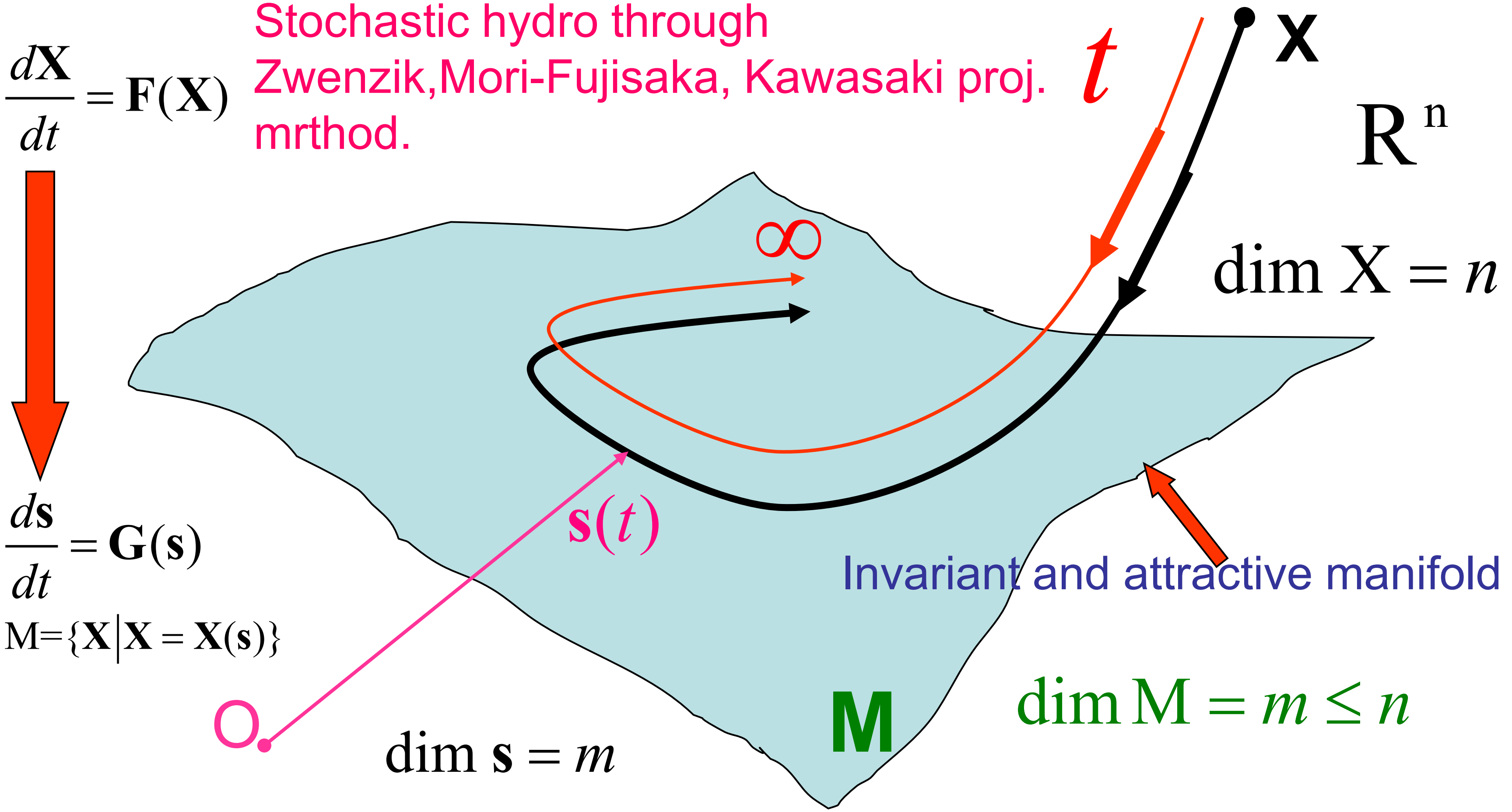
(i) The RG-method incorporating the first fast mode leads to the extended thermodynamics/I-S equation, with new microscopic formulae of the relaxation times.

(ii) On the basis of this development , we propose a new ansatz for the moment method as a rapid reduction

Geometrical image of reduction of dynamics

Stochastic hydro through Zwenzik,Mori-Fujisaka, Kawasaki proj. mrthod.

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X})$$



$$\left\{\begin{array}{l} \dfrac{d\mathbf{s}}{dt} = \mathbf{G}(\mathbf{s}) \\ \mathrm{M}=\{\mathbf{X} \mid \mathbf{X} = \mathbf{X}(\mathbf{s})\} \end{array}\right.$$

$\dim \mathrm{X} = n$

Invariant and attractive manifold

$\dim \mathrm{M} = m \leq n$

$\dim \mathbf{s} = m$

eg.

$\mathbf{X} = f(\mathbf{r},\mathbf{p})$; **distribution function in the phase space (infinite dimensions)**

$\mathbf{s} = \{u^{\mu}, T, n\}$; **the hydrodinamic quantities (5 dimensions), conserved quantities.**

A drawback in the moment method: ambiguity

Boltzmann eq.: $p^\mu\,\partial_\mu f_p(x) = C[f]_p(x)$

$$\partial_\mu N^\mu(x) \;\equiv\; \partial_\mu\left[\sum_p \frac{1}{p^0}\,p^\mu\,f_p(x)\right] = 0, \qquad \partial_\mu T^{\mu\nu}(x) \;\equiv\; \partial_\mu\left[\sum_p \frac{1}{p^0}\,p^\mu\,p^\nu\,f_p(x)\right] = 0,$$

$$\partial_\mu\left[\sum_p \frac{1}{p^0}\,p^\mu\,p^{\nu_1}\,\cdots\,p^{\nu_n}\,f_p(x)\right] = \sum_p \frac{1}{p^0}\,p^{\nu_1}\,\cdots\,p^{\nu_n}\,C[f]_p(x)$$

N-th moment

Making an ansatzs for f_p in a truncated function space,
f_p can be determined in a nonperturbative way.

BUT! In an ambiguous way

$$\partial_\mu\left[\sum_p \frac{1}{p^0}\,p^\mu\,p^{\nu_1}\,\cdots\,p^{\nu_n}\,K_p\,f_p(x)\right] = \sum_p \frac{1}{p^0}\,p^{\nu_1}\,\cdots\,p^{\nu_n}\,K_p\,C[f]_p(x)$$

Arbitrary!

Results from the RG method（Tsumura, Kunihiro, in preparation)

Eg. in the energy rame

$$f_p(x) = f_p^{\rm eq}(x)\,\big(1 + \delta\Phi_p(x)\big), \qquad f_p^{\rm eq}(x) = (2\,\pi)^{-3}\,\exp\left[\frac{\mu(x) - p\cdot u(x)}{T(x)}\right]$$

$$\delta\Phi_p(x) = -\frac{1}{T(x)}\sum_q \mathcal{L}^{-1}_{pq}(x)\,\Big(\Pi_q(x)\,\Pi(x) + J^\mu_q(x)\,J_\mu(x) + \pi^{\mu\nu}_q(x)\,\pi_{\mu\nu}(x)\Big)$$

$$\mathcal{L}_{pq}(x) \equiv f_p^{{\rm eq}-1}(x)\,\frac{\partial}{\partial f_q}C[f]_p\bigg|_{f=f^{\rm eq}(x)} f_q^{\rm eq}(x)$$

$\Pi_p(x),\ J^\mu_p(x),\ \pi^{\mu\nu}_p(x)$ are microscopic dissipative flows:

$$\begin{aligned}
\Pi_p &\equiv \Big(\frac{4}{3} - \gamma\Big)\,(p\cdot u)^2 + \Big((\gamma - 1)\,T\,\hat{h} - \gamma\,T\Big)\,(p\cdot u) - \frac{1}{3}\,m^2,\\
J^\mu_p &\equiv -\Big((p\cdot u) - T\,\hat{h}\Big)\,\Delta^{\mu\nu}\,p_\nu,\\
\pi^{\mu\nu}_p &\equiv \Delta^{\mu\nu\rho\sigma}\,p_\rho\,p_\sigma,
\end{aligned}$$

c.f. Israel-Stewar

$$\delta\Phi_p^{\rm IS}(x) = -\frac{1}{T(x)}\Big(\Pi_p(x)\,\Pi(x) + J^\mu_p(x)\,J_\mu(x) + \pi^{\mu\nu}_p(x)\,\pi_{\mu\nu}(x)\Big)$$

Denicol et al (2010)

$$\delta\Phi_p^{\rm D}(x) = -\frac{1}{T(x)}\,\frac{1}{p\cdot u(x)}\Big(\Pi_p(x)\,\Pi(x) + J^\mu_p(x)\,J_\mu(x) + \pi^{\mu\nu}_p(x)\,\pi_{\mu\nu}(x)\Big)$$

Our formulas:

$$\begin{aligned}
\zeta^{\rm TK} &= -\frac{1}{T}\,\langle\,\tilde{\Pi}\,,\,L^{-1}\,\tilde{\Pi}\,\rangle = -\frac{1}{T}\,\langle\,\Pi\,,\,\mathcal{L}^{-1}\,\Pi\,\rangle_{\rm eq},\\
\lambda^{\rm TK} &= \frac{1}{3\,T^2}\,\langle\,\tilde{J}^\mu\,,\,L^{-1}\,\tilde{J}_\mu\,\rangle = \frac{1}{3\,T^2}\,\langle\,J^\mu\,,\,\mathcal{L}^{-1}\,J_\mu\,\rangle_{\rm eq},\\
\eta^{\rm TK} &= -\frac{1}{10\,T}\,\langle\,\tilde{\pi}^{\mu\nu}\,,\,L^{-1}\,\tilde{\pi}_{\mu\nu}\,\rangle = -\frac{1}{10\,T}\,\langle\,\pi^{\mu\nu}\,,\,\mathcal{L}^{-1}\,\pi_{\mu\nu}\,\rangle_{\rm eq}
\end{aligned}$$

Where,

$$\begin{aligned}
(\tilde{\Pi},\,\tilde{J}^\mu,\,\tilde{\pi}^{\mu\nu}) &= (\Pi,\,J^\mu,\,\pi^{\mu\nu})/(p\cdot u),\\
L_{pq} &= \ \mathcal{L}_{pq}/(p\cdot u).
\end{aligned}$$

$$\langle\,\varphi\,,\,\psi\,\rangle = \textstyle\sum_p \frac{1}{p^0}\,(p\cdot u)\,f_p^{\rm eq}\,\varphi_p\,\psi_p.$$

Results (cont'd)

Relaxation times:

$$\tau_\Pi^{\rm TK} = -\frac{\langle\,\tilde\Pi\,,\,L^{-2}\,\tilde\Pi\,\rangle}{\langle\,\tilde\Pi\,,\,L^{-1}\,\tilde\Pi\,\rangle},$$

$$\tau_J^{\rm TK} = -\frac{\langle\,\tilde J^\mu\,,\,L^{-2}\,\tilde J_\mu\,\rangle}{\langle\,\tilde J^\nu\,,\,L^{-1}\,\tilde J_\nu\,\rangle},$$

$$\tau_\pi^{\rm TK} = -\frac{\langle\,\tilde\pi^{\mu\nu}\,,\,L^{-2}\,\tilde\pi_{\mu\nu}\,\rangle}{\langle\,\tilde\pi^{\rho\sigma}\,,\,L^{-1}\,\tilde\pi_{\rho\sigma}\,\rangle}.$$

In terms of the correlation functions:

Def.

$$R_\zeta(s) \equiv \frac{1}{T}\langle\,\tilde\Pi(0)\,,\,\tilde\Pi(s)\,\rangle,$$

$$R_\lambda(s) \equiv -\frac{1}{3T^2}\langle\,\tilde J^\mu(0)\,,\,\tilde J_\mu(s)\,\rangle$$

$$R_\eta(s) \equiv \frac{1}{10\,T}\langle\,\tilde\pi^{\mu\nu}(0)\,,\,\tilde\pi_{\mu\nu}(s)\,\rangle$$

Then,

$$\zeta^{\rm TK} = \int_0^\infty {\rm d}s\;\;R_\zeta(s),$$

$$\lambda^{\rm TK} = \int_0^\infty {\rm d}s\;\;R_\lambda(s),$$

$$\eta^{\rm TK} = \int_0^\infty {\rm d}s\;\;R_\eta(s),$$

$$\tau_\Pi^{\rm TK} = \frac{\int_0^\infty {\rm d}s\;\;s\,R_\zeta(s)}{\int_0^\infty {\rm d}s\;\;R_\zeta(s)},$$

$$\tau_J^{\rm TK} = \frac{\int_0^\infty {\rm d}s\;\;s\,R_\lambda(s)}{\int_0^\infty {\rm d}s\;\;R_\lambda(s)},$$

$$\tau_\pi^{\rm TK} = \frac{\int_0^\infty {\rm d}s\;\;s\,R_\eta(s)}{\int_0^\infty {\rm d}s\;\;R_\eta(s)}.$$

A natural results!

K. Tsumura and TK, in preparation.

IS and Debicol et al

$$\eta^{\rm IS} = -\frac{1}{10T}\frac{\langle \pi^{ab}, \pi_{ab}\rangle_{\rm eq}\langle \pi^{cd}, \pi_{cd}\rangle_{\rm eq}}{\langle \pi^{ef}, \mathcal{L}\pi_{ef}\rangle_{\rm eq}}, \quad \tau_\pi^{\rm IS} = -\frac{\langle \pi^{\mu\nu}, \pi_{\mu\nu}\rangle}{\langle \pi^{\rho\sigma}, \mathcal{L}\pi_{\rho\sigma}\rangle_{\rm eq}},$$

← Israel－Stewart

$$\eta^{\rm D} = -\frac{1}{10T}\frac{\langle \tilde{\pi}^{ab}, \pi_{ab}\rangle_{\rm eq}\langle \pi^{cd}, \tilde{\pi}_{cd}\rangle_{\rm eq}}{\langle \tilde{\pi}^{ef}, \mathcal{L}\tilde{\pi}_{ef}\rangle_{\rm eq}}, \quad \tau_\pi^{\rm D} = -\frac{\langle \tilde{\pi}^{\mu\nu}, \tilde{\pi}_{\mu\nu}\rangle}{\langle \tilde{\pi}^{\rho\sigma}, \mathcal{L}\tilde{\pi}_{\rho\sigma}\rangle_{\rm eq}},$$

← Denicol et al

both of which do not include the second and higer order terms in the coll. op.

$$\mathcal{L}_{pq} = (p\cdot u)\,L_{pq}$$

The ratios of rel. time and transport coeff.:

$$\beta_\pi^{\rm TK} \equiv \frac{\eta^{\rm TK}}{\tau_\pi^{\rm TK}} = \frac{1}{10T}\frac{\langle \tilde{\pi}^{ab}, L^{-1}\tilde{\pi}_{ab}\rangle\langle \tilde{\pi}^{cd}, L^{-1}\tilde{\pi}_{cd}\rangle}{\langle \tilde{\pi}^{ef}, L^{-2}\tilde{\pi}_{ef}\rangle},$$

$$\beta_\pi^{\rm IS} \equiv \frac{\eta^{\rm IS}}{\tau_\pi^{\rm IS}} = \frac{1}{10T}\frac{\langle \pi^{ab}, \pi_{ab}\rangle_{\rm eq}\langle \pi^{cd}, \pi_{cd}\rangle_{\rm eq}}{\langle \pi^{ef}, \pi_{ef}\rangle},$$

$$\beta_\pi^{\rm D} \equiv \frac{\eta^{\rm D}}{\tau_\pi^{\rm D}} = \frac{1}{10T}\langle \tilde{\pi}^{\mu\nu}, \tilde{\pi}_{\mu\nu}\rangle,$$

If the mom. dep. of the crosssection is neglegible, Denicol will be fine.

⇨ Ritz－Galerkin approx. Is valid. Then ⇨

$$\beta_\pi^{\rm TK} \sim \frac{1}{10T}\langle \tilde{\pi}^{\mu\nu}, \tilde{\pi}_{\mu\nu}\rangle = \beta_\pi^{\rm D} \neq \beta_\pi^{\rm IS}.$$

Denicol et al formulae OK
But I-S not.

Brief summary

- The RG method was used to derive covariant rel. diss. Hydro. Eq. in a generic frame.
- Our equaions ensure the stability of the thermal eq. state.
- We extended to the case of the second order.
- We proposed a new ansatz for Maxwell-Grad moment method on the basis of the RG results.
- We have clarified the approximate nature of IS and Denicol et al formulae.

Back Ups

Basics about rel. hydrodynamics

1. The fluid dynamic equations as conservation (balance) equations

$$\partial_\mu N_i^\mu \equiv 0\ ,\quad i = 1, \ldots, n\ ,$$ local conservation of charges

$$\partial_\mu T^{\mu\nu} \equiv 0\ ,\quad \nu = 0, \ldots, 3\ .$$ local conservation of energy-mom.

2.Tensor decomposition and choice of frame

u^μ ; **arbitrary** normalized **time-like** vector $u \cdot u = 1$

Def. space-like projection $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu\ ,\ \Delta^{\mu\nu} u_\nu = 0\ ,\ \Delta^{\mu\alpha}\Delta_\alpha^\nu = \Delta^{\mu\nu}$

space-like vector

space-like traceless tensor

$$N_i^\mu = n_i\, u^\mu + \nu_i^\mu\ ,$$

$$T^{\mu\nu} = \epsilon\, u^\mu u^\nu - p\, \Delta^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu + \pi^{\mu\nu}$$

$n_i \equiv N_i \cdot u$; net density of charge *i* in the **Local Rest Frame**

$\nu_i^\mu \equiv \Delta_\nu^\mu N_i^\nu$; net flow in LRF

$\epsilon \equiv u_\mu T^{\mu\nu} u_\nu$; energy density in LRF $p \equiv -\frac{1}{3} T^{\mu\nu}\Delta_{\mu\nu}$; isotropic pressure in LRF

$q^\mu \equiv \Delta^{\mu\alpha} T_{\alpha\beta} u^\beta$; heat flow in LRF

$$\pi^{\mu\nu} \equiv \left[\frac{1}{2}\left(\Delta_\alpha^\mu \Delta_\beta^\nu + \Delta_\beta^\mu \Delta_\alpha^\nu\right) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}\right] T^{\alpha\beta}$$; stress tensor in LRF

Grad-Mueller type eq.

$$\begin{aligned} \frac{\partial}{\partial t} n + \boldsymbol{\nabla} \cdot (n\, \boldsymbol{u}) &= 0, \\ m\, n\, \frac{\partial}{\partial t} u^i + m\, n\, \boldsymbol{u} \cdot \boldsymbol{\nabla} u^i &= -\nabla^j (p\, \delta^{ji} + 2\, \eta\, \pi^{ji}), \\ n\, \frac{\partial}{\partial t} e + n\, \boldsymbol{u} \cdot \boldsymbol{\nabla} e &= -\nabla^j (T\, \lambda\, J^j) - 2\, \eta\, \pi^{jk}\, \bar{X}_{\pi}^{jk} - p\, \boldsymbol{\nabla} \cdot \boldsymbol{u}, \end{aligned}$$

$p = n\,T$ と $e = 3/2\,T$

$$\begin{aligned} \pi^{ij} + \tau_\pi \Big(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \Big) \pi^{ij} + \ell_{\pi J}\, \Delta^{ijmk}\, \nabla^m J^k &= -\bar{X}_{\pi}^{ij} + \bar{X}_{\pi\pi}^{ijkl}\, \pi^{kl} + \bar{X}_{\pi J}^{ijk}\, J^k, \\ J^i + \tau_J \Big(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \Big) J^i + \ell_{J\pi}\, \nabla^m \pi^{mi} &= -\bar{X}_{J}^{i} + \bar{X}_{J\pi}^{ikl}\, \pi^{kl} + \bar{X}_{JJ}^{ik}\, J^k, \end{aligned}$$

$$\begin{aligned} \bar{X}_{\pi\pi}^{ijkl} \equiv -\frac{T}{4\,\eta} \Bigg\{ \Bigg[& \frac{\partial}{\partial t} \Big(\frac{2\,\eta\,\tau_\pi}{T} \Big) + \boldsymbol{\nabla} \cdot \Big(\frac{2\,\eta\,\tau_\pi}{T}\, \boldsymbol{u} \Big) \\ & + A_{\pi\pi}^{(n)} \frac{1}{n} \Big(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \Big) n + A_{\pi\pi}^{(T)} \frac{1}{T} \Big(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \Big) T + A_{\pi\pi}^{(u,1)}\, \boldsymbol{\nabla} \cdot \boldsymbol{u} \Bigg] \Delta^{ijkl} \\ & + A_{\pi\pi}^{(u,2)}\, \Delta^{ijma}\, \Delta^{ankl}\, \bar{X}_{\pi}^{mn} + B_{\pi\pi}^{(u)}\, 2\, \Delta^{ijma}\, \Delta^{ankl}\, \omega^{mn} \Bigg\}, \end{aligned}$$

with the vorticity,

$$\omega^{mn} \equiv \frac{1}{2} (\nabla^m u^n - \nabla^n u^m),$$

etc.

$$\bar{X}^{ijk}_{\pi J} \;\equiv\; -\frac{T}{4\,\eta}\Bigg\{\Bigg[\nabla^m\Big(\frac{2\,\eta\,\ell_{\pi J}}{T}\Big) + (A^{(u)}_{\pi J}+B^{(u)}_{\pi J})\Big(\frac{\partial}{\partial t}+\boldsymbol{u}\cdot\boldsymbol{\nabla}\Big)u^m + (A^{(n)}_{\pi J}+B^{(n)}_{\pi J})\,\frac{1}{n}\,\nabla^m n + (A^{(T)}_{\pi J}+B^{(T)}_{\pi J})\,\frac{1}{T}\,\nabla^m T\Bigg]\,\Delta^{ijmk}\Bigg\}, \tag{IV.141}$$

$$\bar{X}^{ikl}_{J\pi} \;\equiv\; -\frac{1}{2\,\lambda}\Bigg\{\Bigg[\nabla^m\Big(\lambda\,\ell_{J\pi}\Big) + (A^{(u)}_{J\pi}+B^{(u)}_{J\pi})\Big(\frac{\partial}{\partial t}+\boldsymbol{u}\cdot\boldsymbol{\nabla}\Big)u^m + (A^{(n)}_{J\pi}+B^{(n)}_{J\pi})\,\frac{1}{n}\,\nabla^m n + (A^{(T)}_{J\pi}+B^{(T)}_{J\pi})\,\frac{1}{T}\,\nabla^m T\Bigg]\,\Delta^{imkl}\Bigg\}, \tag{IV.142}$$

$$\bar{X}^{ik}_{JJ} \;\equiv\; -\frac{1}{2\,\lambda}\Bigg\{\Bigg[\frac{\partial}{\partial t}\Big(\lambda\,\tau_J\Big) + \boldsymbol{\nabla}\cdot\Big(\lambda\,\tau_J\,\boldsymbol{u}\Big) + A^{(n)}_{JJ}\,\frac{1}{n}\,\Big(\frac{\partial}{\partial t}+\boldsymbol{u}\cdot\boldsymbol{\nabla}\Big)n + A^{(T)}_{JJ}\,\frac{1}{T}\,\Big(\frac{\partial}{\partial t}+\boldsymbol{u}\cdot\boldsymbol{\nabla}\Big)T + A^{(u,1)}_{JJ}\,\boldsymbol{\nabla}\cdot\boldsymbol{u}\Bigg]\,\Delta^{ik} + A^{(u,2)}_{JJ}\,\bar{X}^{ik}_{\pi} + B^{(u)}_{JJ}\,2\,\omega^{ik}\Bigg\}, \tag{IV.143}$$

$$\begin{aligned}
\tau_\pi &\equiv \frac{T}{10\,\eta}\,\langle\,\tilde{\pi}^{ab}\,,\,\tilde{\pi}^{ab}\,\rangle = \frac{1}{10\,T\,\eta}\,\langle\,f^{\mathrm{eq}}\,\hat{\pi}^{ab}\,,\,A^{-2}\,f^{\mathrm{eq}}\,\hat{\pi}^{ab}\,\rangle,\\
\tau_J &\equiv \frac{1}{3\,\lambda}\,\langle\,\tilde{J}^{a}\,,\,\tilde{J}^{a}\,\rangle = \frac{1}{3\,T^2\,\lambda}\,\langle\,f^{\mathrm{eq}}\,\hat{J}^{a}\,,\,A^{-2}\,f^{\mathrm{eq}}\,\hat{J}^{a}\,\rangle,\\
\ell_{\pi J} &\equiv \frac{T}{10\,\eta}\,\langle\,\tilde{\pi}^{ab}\,,\,\delta v^a\,\tilde{J}^{b}\,\rangle = \frac{1}{10\,T\,\eta}\,\langle\,f^{\mathrm{eq}}\,\tilde{\pi}^{ab}\,,\,A^{-1}\,\delta v^a\,A^{-1}\,f^{\mathrm{eq}}\,\tilde{J}^{b}\,\rangle,\\
\ell_{J\pi} &\equiv \frac{1}{3\,\lambda}\,\langle\,\tilde{J}^{a}\,,\,\delta v^b\,\tilde{\pi}^{ab}\,\rangle = \frac{1}{3\,T^2\,\lambda}\,\langle\,f^{\mathrm{eq}}\,\tilde{J}^{a}\,,\,A^{-1}\,\delta v^b\,A^{-1}\,f^{\mathrm{eq}}\,\tilde{\pi}^{ab}\,\rangle.
\end{aligned}$$

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