Correlation function and response function in shell model of turbulence

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Fluctuation response relation

- $X(t)$: quantity of some statistically steady-state system (many degrees of freedom)
  - Auto correlation function ($\langle \quad \rangle$: ensemble average).

$$C(t-s) = \langle X(t)X(s) \rangle$$

  - Response function to fluctuation $f(t)$

$$G(t-s) = \left\langle \frac{\delta X(t)}{\delta f(s)} \right\rangle$$

In an integral form, $X(t) + \delta X(t) = X(t) + \int_0^t G(t-s) f(s) ds$.

- Fluctuation response relation (fluctuation dissipation relation)

$$G(t-s) = \beta \, C(t-s)$$

(β: inverse temperature in equilibrium systems).

- Formal expression of the response function

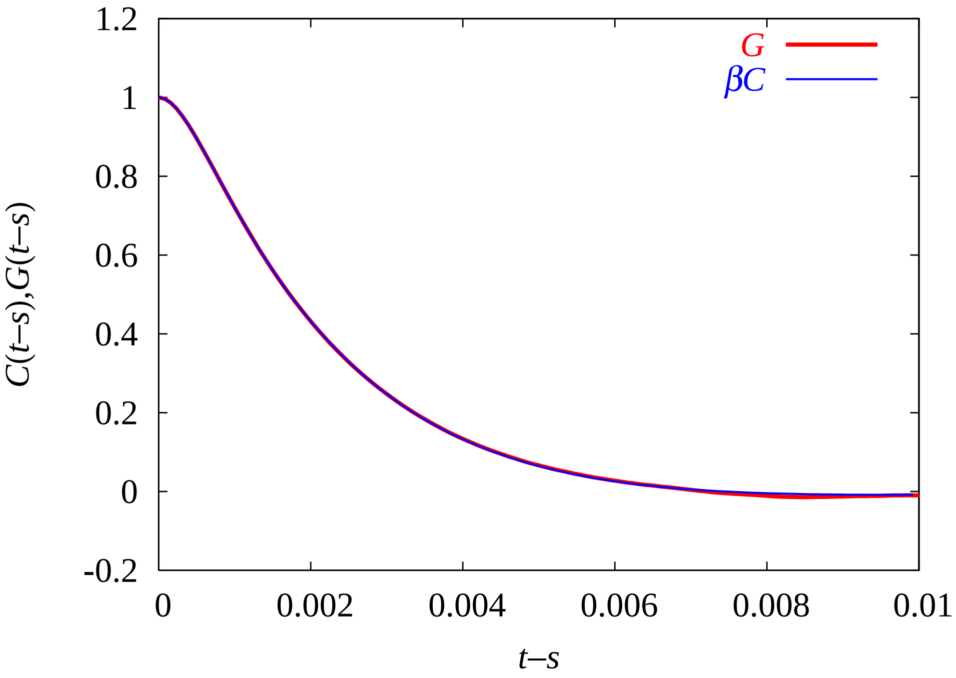
$$G(t-s) = -\left\langle X(t) \left. \frac{\partial \ln \rho(X,t)}{\partial X} \right|_{t=s} \right\rangle$$

$\rho(X,t)$: probability distribution function of X.

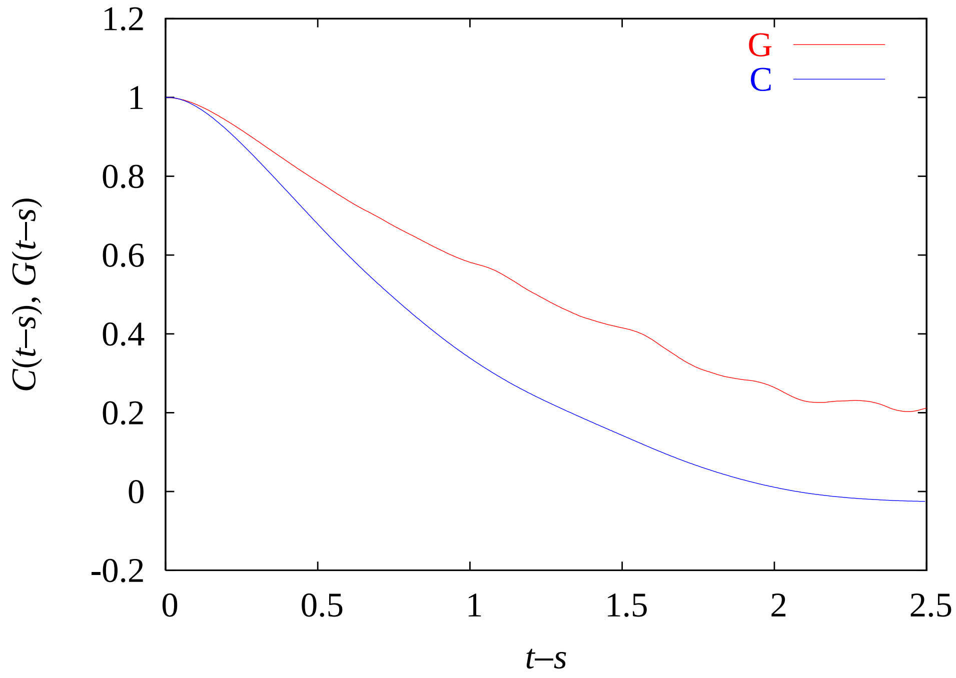
If the distribution ρ is not Gaussian, $G \not\propto C$ in general!

Typical examples

- Correlation function $C(t-s) = \langle X(t)X(s) \rangle$
  Response function $G(t-s) = \left\langle \frac{\delta X(t)}{\delta f(s)} \right\rangle$



Gaussian system $G \propto C$



non-Gaussian system $G \not\propto C$

Implication to statistical theory of turbulence

- Incompressible Navier-Stokes eq. with forcing

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\nabla p + \nu \nabla^2 \boldsymbol{u} + \boldsymbol{f}, \quad \nabla \cdot \boldsymbol{u} = 0. \tag{1}$$

- Expression in the Fourier space $[\boldsymbol{u}(\boldsymbol{x}, t) = \sum_{\boldsymbol{k}} \hat{\boldsymbol{u}}(\boldsymbol{k}, t) \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}}]$

$$\partial_t \hat{u}_j(\boldsymbol{k}, t) = -\frac{\mathrm{i}}{2} \sum_{l,m=1}^{3} P_{jlm}(\boldsymbol{k}) \sum_{\substack{\boldsymbol{p},\boldsymbol{q} \\ \boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{o}}} \hat{u}_l(-\boldsymbol{p}, t) \hat{u}_m(-\boldsymbol{q}, t) - \nu |\boldsymbol{k}|^2 \hat{u}_j(\boldsymbol{k}, t) + \hat{f}_j(\boldsymbol{k}, t) \tag{2}$$

Holy grail: closure eq. of the correlation func. $C_{jl}(\boldsymbol{k}, t, s) = \langle \hat{u}_j(\boldsymbol{k}, t) \hat{u}_l(-\boldsymbol{k}, s) \rangle$

- Direct interaction approximation (DIA), (R.H. Kraichnan 1959)
  - Decompose $\hat{u} \Rightarrow \hat{u} + \delta\hat{u}$ and get the linearized eq. of $\delta\hat{u}$ from (2).
  - Response func. of the linearized eq.: $G_{jl}(\boldsymbol{k}, t, s) = \left\langle \dfrac{\delta \hat{u}_j(\boldsymbol{k}, t)}{\delta \hat{u}_l(-\boldsymbol{k}, s)} \right\rangle$
  - Closure eqs. of C and G (under statistical homogeneity and isotropy):
    $[\partial_t + \nu k^2 + F_1(C, k, t, t')]C(k, t, t') = 0,$
    $[\partial_t + \nu k^2 + F_1(C, k, t, t')]G(k, t, t') = 0,$
    $(\partial_t + 2\nu k^2)C(k, t, t') =$
    $\int dk' \int dk'' F_2(k, k', k'') \int_{-\infty}^{t} ds C(k', t, t')[G(k, t, t')C(k'', t, t') - G(k', t, t')C(k, t, t')]$
  - **These closure eqs. are solved by (naturally) assuming $G \propto C$.**

- Our final goal is G and C of turbulence but....
- **Problem**: calculation of the response function $G_{jl}(\boldsymbol{k},t,s) = \left\langle \dfrac{\delta \hat{u}_j(\boldsymbol{k},t)}{\delta \hat{u}_l(-\boldsymbol{k},s)} \right\rangle$

  is numerically costly !
- Less-costly models of turbulence
  - Turbulence in two dimensions
  - Burgers equation (Burgers turbulence) $\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \nu\nabla^2\boldsymbol{u}.$
  - "shell models" of turbulence
  - $\ldots$
    - shell models (reduced dynamical-system models)
      - homogeneous, isotropic turbulence (Obukhov 1971; Gledzer 1973; Yamada & Ohkitani 1987).
      - thermal convection turbulence (Suzuki & Toh 1995).
      - magnetohydrodynamic (MHD) turbulence (Hattori & Ishizawa 2001).
      - quantum turbulence (Wacks & Barenghi 2011).
      - $\ldots$

Dynamical-system model of turbulence: shell model

- Gledzer-Ohkitani-Yamada shell model (Yamada & Ohkitani 1987)

$$\left(\frac{d}{dt} + \nu k_j^2\right) u_j(t) = \mathrm{i}[k_j u_{j+2} u_{j+1}^* - \tfrac{1}{2} k_{j-1} u_{j+1} u_{j-1}^* - \tfrac{1}{2} k_{j-2} u_{j-1} u_{j-2}] + f_j$$

$u_j(t) \in \mathbb{C}$; $j = 1, \ldots, N$; $k_j = k_0 2^j$; $\cdot^*$ complex conjugate.

  – "shell" : annulus in the wavenumber space $2^j \leq |\boldsymbol{k}| \leq 2^{j+1}$
  – This is a minimalistic model of the Navier-Stokes eq. in the Fourier space

$$\boldsymbol{u}(\boldsymbol{x}, t) = \sum_{\boldsymbol{k}} \hat{\boldsymbol{u}}(\boldsymbol{k}, t) \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}},$$

$$(\partial_t + \nu|\boldsymbol{k}|^2)\hat{u}_n(\boldsymbol{k}, t) = -\frac{\mathrm{i}}{2} \sum_{l,m=1}^{3} P_{nlm}(\boldsymbol{k}) \sum_{\substack{\boldsymbol{p},\boldsymbol{q} \\ \boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{o}}} \hat{u}_l(-\boldsymbol{p}, t)\hat{u}_m(-\boldsymbol{q}, t) + \hat{f}_n(\boldsymbol{k}, t).$$

$u_j(t)$ of k_j is a representative of $\hat{\boldsymbol{u}}(\boldsymbol{k}, t)$ in the j-th shell $k_j \leq |\boldsymbol{k}| \leq k_{j+1}$.

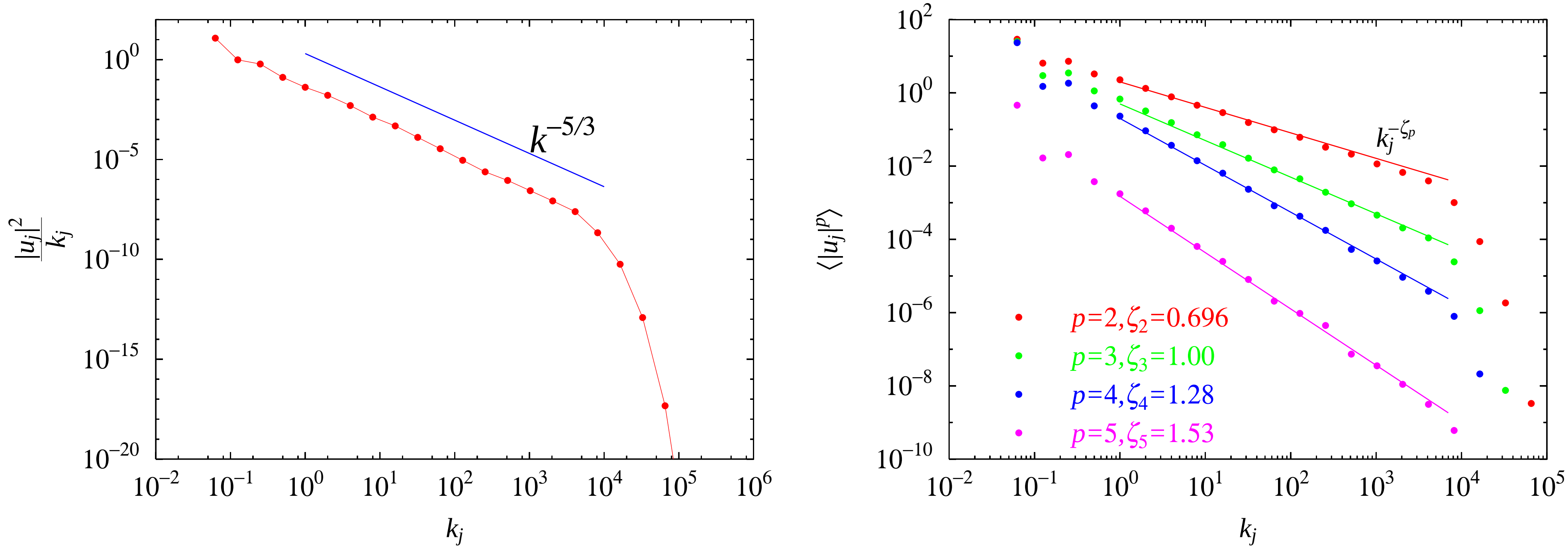
- This Gledzer-Ohkitani-Yamada shell model has a lot of success.

- **Success of the Gledzer-Ohkitani-Yamada shell model**

$$\left(\frac{d}{dt} + \nu k_j^2\right) u_j(t) = \mathrm{i}[k_j u_{j+2} u_{j+1}^* - \tfrac{1}{2} k_{j-1} u_{j+1} u_{j-1}^* - \tfrac{1}{2} k_{j-2} u_{j-1} u_{j-2}] + f_j$$

$$(u_j(t) \in \mathbb{C}; \quad j = 1, \ldots, 24; \quad k_j = k_0 2^j; \quad f_j \text{ is zero except } f_1 = 0.5 + 0.5\mathrm{i}).$$

The shell model can reproduce some characteristics of the Navier-Stokes turbulence.



Energy spectrum $\frac{|u_j|^2}{k_j}$ $\qquad$ p-th order moments $\langle |u_j|^p \rangle \propto k_j^{-\zeta_p}$

- Scaling exponents ζ_p of the moments coincides with those of the NS turbulence

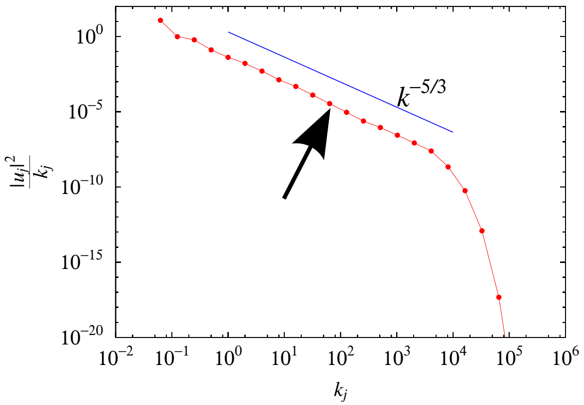
$$\left\langle \left\{ [\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{r})] \cdot \frac{\boldsymbol{r}}{|\boldsymbol{r}|} \right\}^p \right\rangle \propto |\boldsymbol{r}|^{\zeta_p}.$$

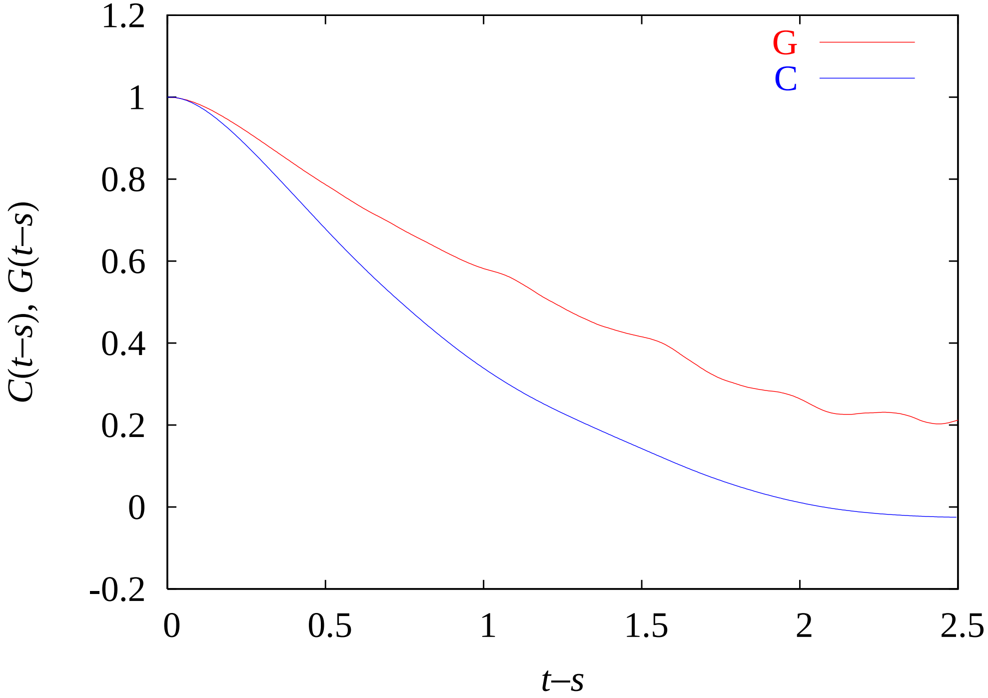
Correlation and response functions of the shell model

$$\left(\frac{d}{dt} + \nu k_j^2\right) u_j(t) = \mathrm{i}[k_j u_{j+2} u_{j+1}^* - \tfrac{1}{2} k_{j-1} u_{j+1} u_{j-1}^* - \tfrac{1}{2} k_{j-2} u_{j-1} u_{j-2}] + f_j$$

$$(u_j(t) \in \mathbb{C}; \quad j = 1, \ldots, 24; \quad k_j = k_0 2^j; \quad f_j \text{ is zero except } f_1 = 0.5 + 0.5\mathrm{i})$$

- Correlation and response functions in the inertial range

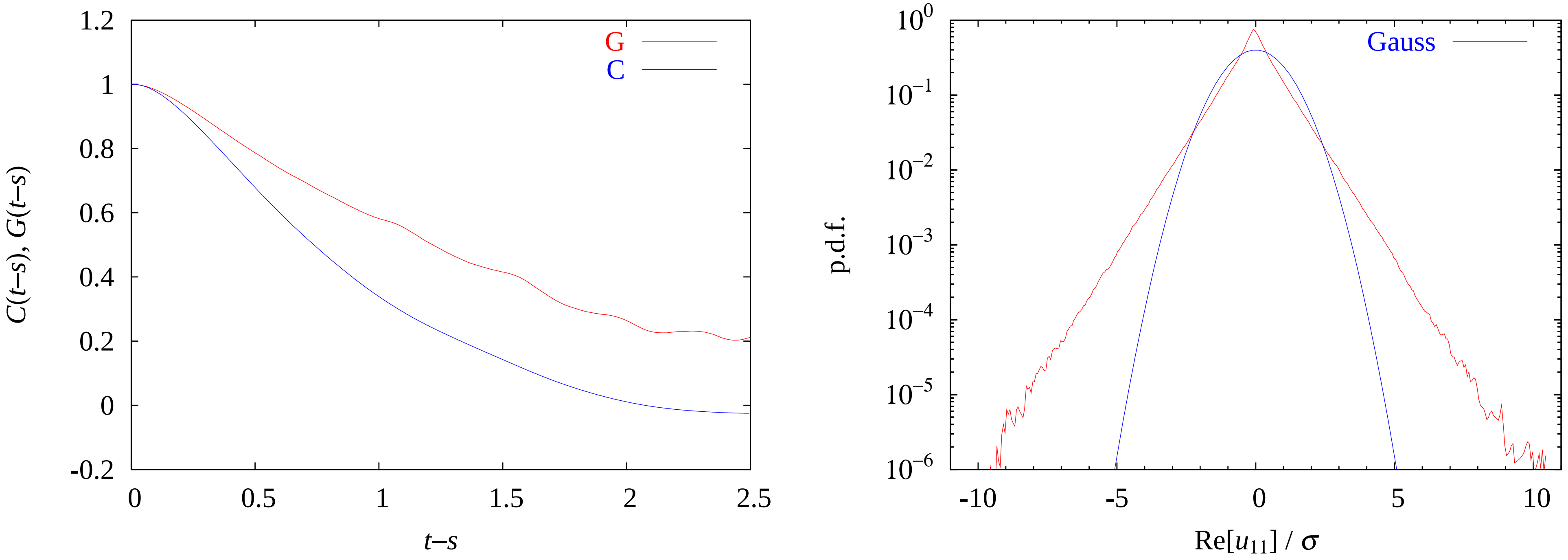




Response function $\quad G(t-s) = \mathrm{Re}\left\langle \frac{\delta u_{11}(t)}{\delta u_{11}(s)} \right\rangle$ ($\delta u_{11}(s)$ is purely real.)

Auto correlation function $C(t-s) = \frac{\langle \mathrm{Re}[u_{11}(t)]\ \mathrm{Re}[u_{11}(s)]\rangle}{\langle \{\mathrm{Re}[u_{11}(t)]\}^2 \rangle}$ (normalized)

- The fluctuation-dissipation relation $G \propto C$ breaks down for the shell model (Biferale *et al.* 2002).



- An expression for this discrepancy between G and C for the shell model?

After trial and error, we find :

a relation between G and C can be obtained if we add noises to the shell model

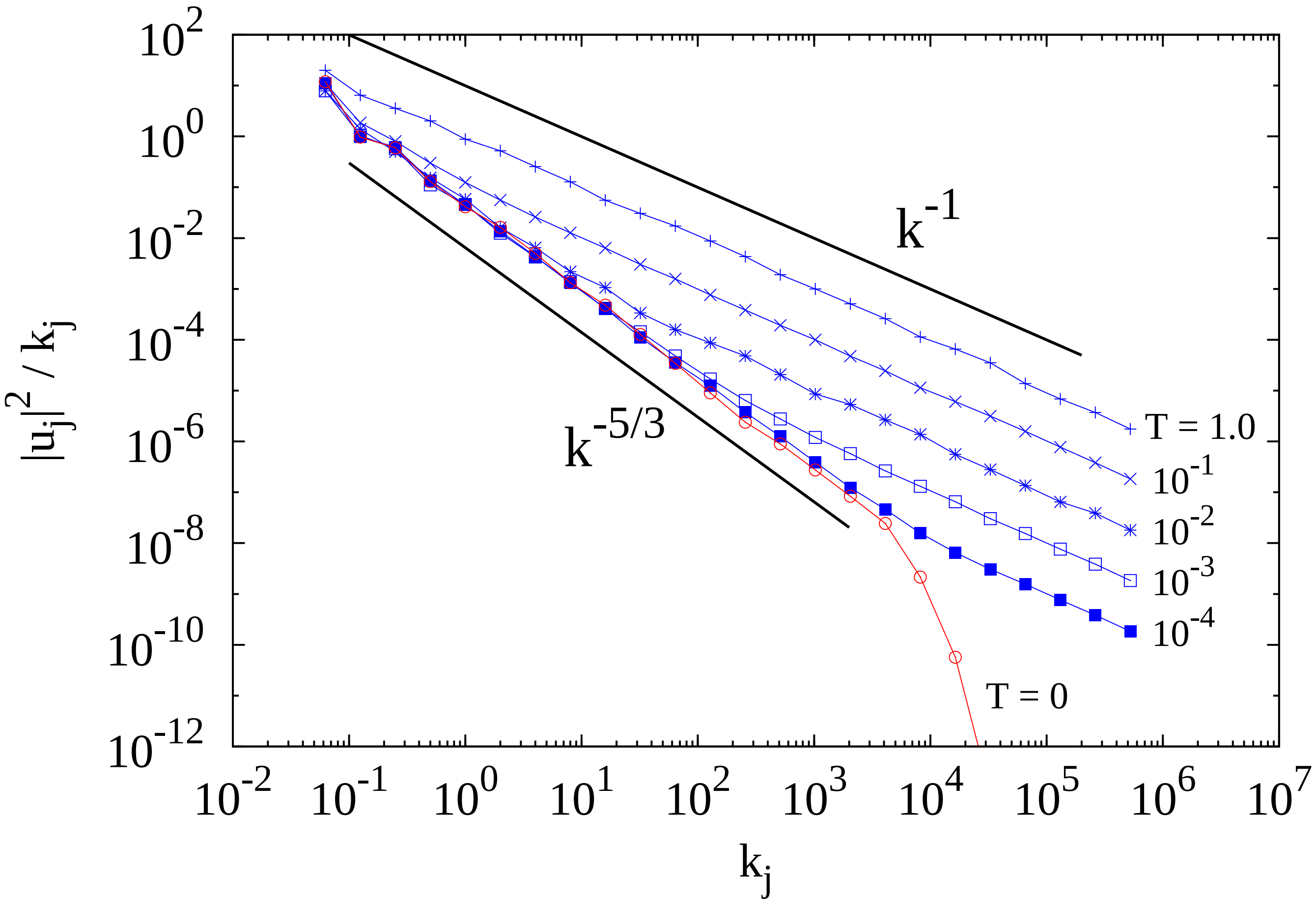
$$\frac{d}{dt}u_j = \mathrm{i}[k_j u_{j+2} u^*_{j+1} - \frac{1}{2}k_{j-1}u_{j+1}u^*_{j-1} - \frac{1}{2}k_{j-2}u_{j-1}u_{j-2}] + f_j - \nu k_j^2 u_j + \xi_j.$$

ξ_j: Gaussian white noise: $\langle \xi_j(t) \rangle = 0, \ \ \langle \xi_j(t) \xi^*_\ell(s) \rangle = 2\nu k_j^2 \, T \, \delta_{j,\ell} \, \delta(t-s).$

Property of the shell model with noise

$$\frac{d}{dt}u_j = \mathrm{i}[k_j u_{j+2} u_{j+1}^* - \frac{1}{2}k_{j-1}u_{j+1}u_{j-1}^* - \frac{1}{2}k_{j-2}u_{j-1}u_{j-2}] - \nu k_j^2 u_j + f_j + \xi_j,$$

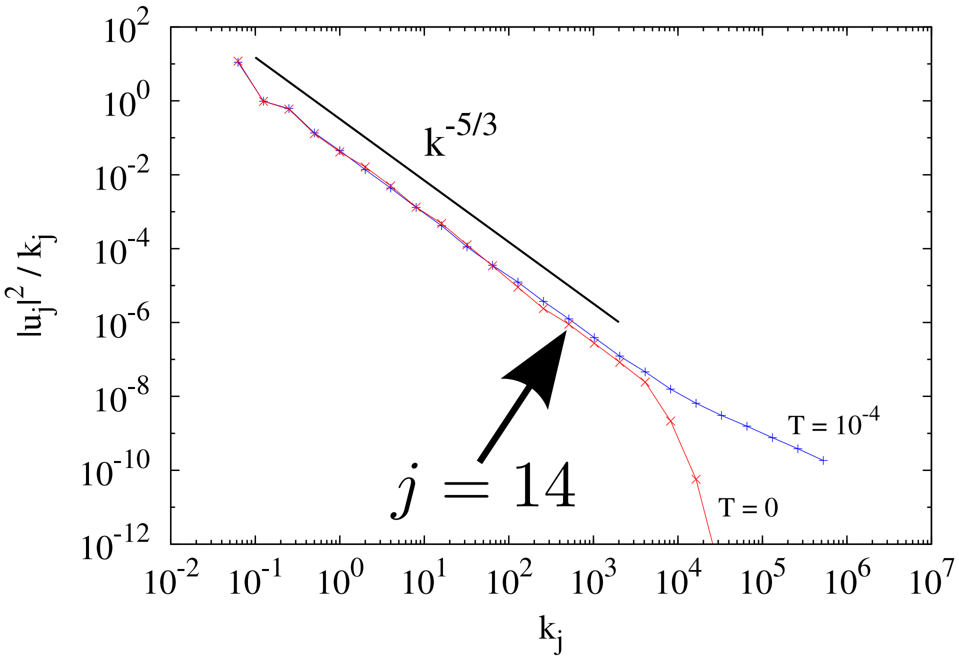
$$\langle \xi_j(t) \xi_\ell^*(s) \rangle = 2\nu k_j^2\, T\, \delta_{j,\ell}\, \delta(t-s).$$

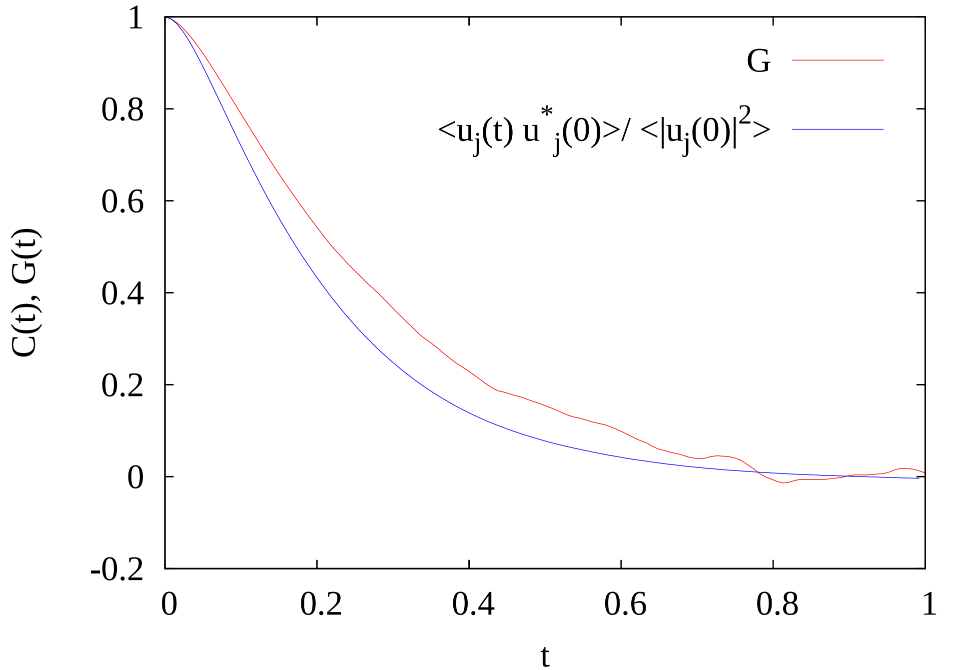


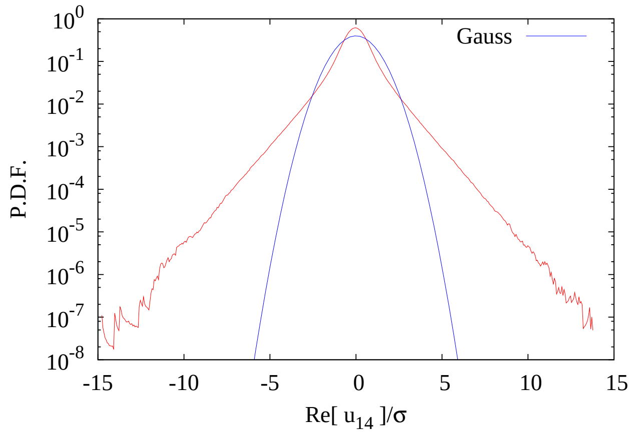
If "the heat bath temperature" T is small enough,
the noisy model is close to the Navier-Stokes turbulence.

The noisy shell model: correlation and response functions $C \not\propto G$

$$\frac{d}{dt}u_j = \mathrm{i}[k_j u_{j+2}u^*_{j+1} - \tfrac{1}{2}k_{j-1}u_{j+1}u^*_{j-1} - \tfrac{1}{2}k_{j-2}u_{j-1}u_{j-2}] - \nu k_j^2 u_j + f_j + \xi_j,$$
$$\langle \xi_j(t)\xi^*_\ell(s)\rangle = 2\nu k_j^2\, T\, \delta_{j,\ell}\, \delta(t-s).$$



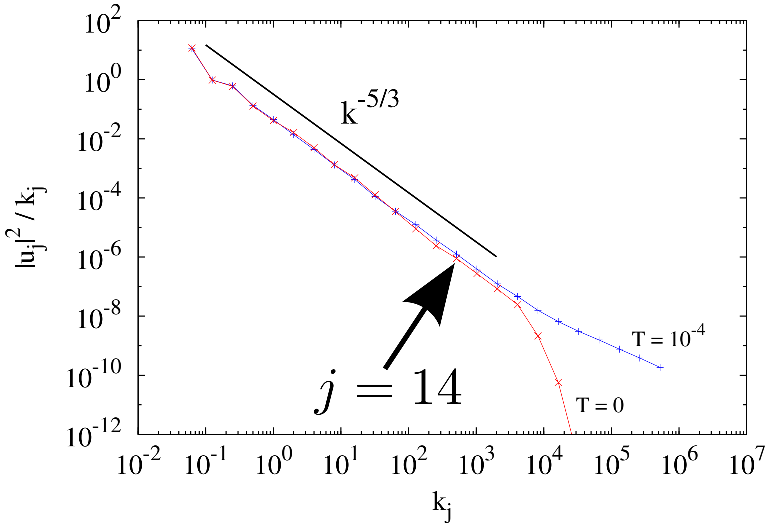


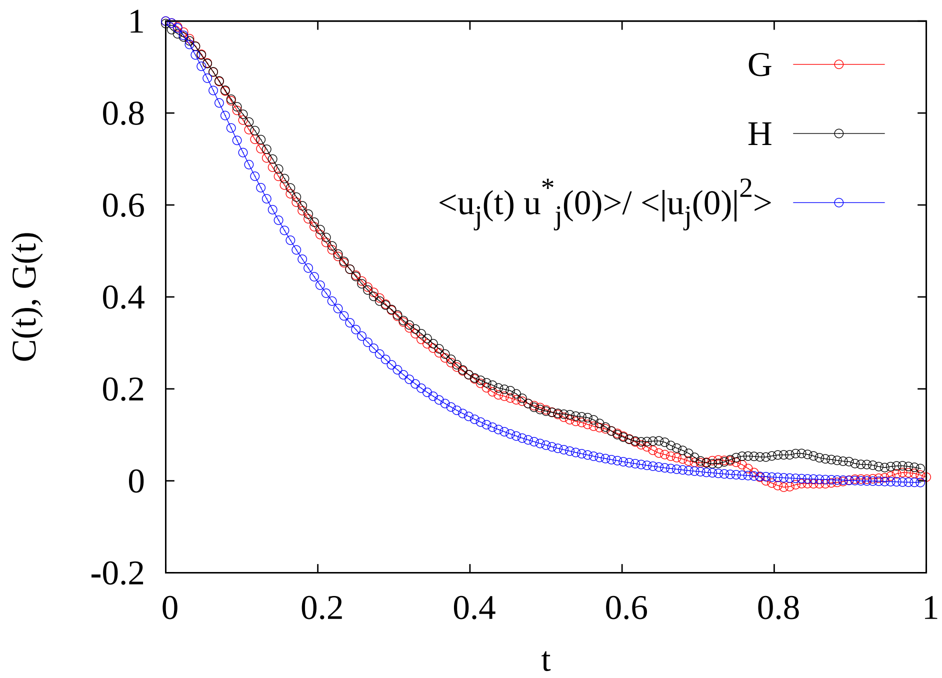


Noisy shell model: Correlation function C and response function G

$$\frac{d}{dt}u_j = \mathrm{i}[k_j u_{j+2} u^*_{j+1} - \tfrac{1}{2}k_{j-1}u_{j+1}u^*_{j-1} - \tfrac{1}{2}k_{j-2}u_{j-1}u_{j-2}] - \nu k_j^2 u_j + f_j + \xi_j,$$

$$\langle \xi_j(t)\xi^*_\ell(s)\rangle = 2\nu k_j^2\, T\, \delta_{j,\ell}\, \delta(t-s).$$





$$G_{jj}(t) = \frac{\delta u_j(t)}{\delta u_j(0)}, \qquad C_{jj}(t) = \langle u_j(t) u_j^*(0)\rangle,$$

$$H_{jj}(t) = \frac{1}{T}C_{jj}(t) - \frac{1}{2\nu k_j^2 T}\left[\langle u_j(t)\Lambda_j^*(0)\rangle + \langle u_j(0)\Lambda_j^*(t)\rangle\right].$$

$$(\Lambda_j(t) = \mathrm{i}[k_j u_{j+2}u^*_{j+1} - \tfrac{1}{2}k_{j-1}u_{j+1}u^*_{j-1} - \tfrac{1}{2}k_{j-2}u_{j-1}u_{j-2}] + f_j).$$

Noisy shell model: correlation function C and response function G

$$\frac{d}{dt}u_j = \underbrace{\mathrm{i}[k_j u_{j+2}u_{j+1}^* - \tfrac{1}{2}k_{j-1}u_{j+1}u_{j-1}^* - \tfrac{1}{2}k_{j-2}u_{j-1}u_{j-2}] + f_j}_{\Lambda_j} - \nu k_j^2 u_j + \xi_j,$$

$$\langle \xi_j(t)\xi_\ell^*(s)\rangle = 2\nu k_j^2\, T\, \delta_{j,\ell}\, \delta(t-s).$$

- The response function is expressed with triple correlations

$$G_{jj}(t-s) \stackrel{\mathrm{def}}{=} \frac{\delta u_j(t)}{\delta u_j(s)} = \frac{1}{T}C_{jj}(t-s) - \frac{1}{2\nu k_j^2 T}\left[\langle u_j(t)\Lambda_j^*(s)\rangle + \langle u_j(s)\Lambda_j^*(t)\rangle\right].$$

- Derivation (Harada & Sasa 2006):

$$\langle u_j(t)\rangle = \int du_0 \int [d\boldsymbol{u}]\rho_0(u_0)\mathcal{T}[\boldsymbol{u}|\boldsymbol{u}_0(t_0)]u_j$$

with the assumption that the transition probability $\mathcal{T}[\boldsymbol{u}|\boldsymbol{u}_0(t_0)]$ is determined by the noise

$$[d\boldsymbol{u}]\mathcal{T}[\boldsymbol{u}|\boldsymbol{u}_0(t_0)] \propto [d\boldsymbol{\xi}]\exp\left[-\frac{1}{2}\sum_{j=1}^{N}\int_{t_0}^{t} ds\frac{|\xi_j(s)|^2}{\sigma_j^2}\right]$$

$$(\sigma_j = \nu k_j^2 T)$$

Summary and outlook

- In non-Gaussian systems, the correlation C and response functions G: $C \not\propto G$.
- What about fluid turbulence? Expression of the discrepancy?
- The Gledzer-Ohkitani-Yamada shell model: a dynamical-system model of turbulence
- In the shell model, $C \not\propto G$ as expected.
- The shell model with noise: again $C \not\propto G$

$$\partial_t u_j = \underbrace{\mathrm{i}[k_j u_{j+2} u^*_{j+1} - \tfrac{1}{2} k_{j-1} u_{j+1} u^*_{j-1} - \tfrac{1}{2} k_{j-2} u_{j-1} u_{j-2}] + f_j}_{\Lambda_j} - \nu k_j^2 u_j + \xi_j,$$

$$\langle \xi_j(t) \xi^*_\ell(s) \rangle = 2\nu k_j^2\, T\, \delta_{j,\ell}\, \delta(t-s), \qquad (k_j = k_0 2^j,\ u_j \in \mathbb{C}).$$

- For the noisy shell model, the expression between G and C $\ (C_{jj}(t) = \langle u_j(t) u_j^*(0) \rangle)$

$$G_{jj}(t-s) = \frac{\delta u_j(t)}{\delta u_j(s)} = \frac{1}{T} C_{jj}(t-s) - \frac{1}{2\nu k_j^2 T} \left[\langle u_j(t) \Lambda_j^*(s) \rangle + \langle u_j(s) \Lambda_j^*(t) \rangle \right].$$

♠ $\langle u_j(t) \Lambda_j^*(s) \rangle$ resembles the energy transfer among the shell.

♣ How about the (noisy?) Navier-Stokes turbulence???