Correlation function and response function in shell model of turbulence

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**Fluctuation response relation**

- $X(t)$: quantity of some statistically steady-state system (many degrees of freedom)
  - Auto correlation function (⟨ ⟩: ensemble average).
    \[
    C(t - s) = \langle X(t)X(s) \rangle
    \]
  - Response function to fluctuation $f(t)$
    \[
    G(t - s) = \left\langle \frac{\delta X(t)}{\delta f(s)} \right\rangle
    \]
    In an integral form, $X(t) + \delta X(t) = X(t) + \int_0^t G(t - s)f(s)ds$.

- Fluctuation response relation (fluctuation dissipation relation)
  \[
  G(t - s) = \beta C(t - s)
  \]
  ($\beta$: inverse temperature in equilibrium systems).

- Formal expression of the response function
  \[
  G(t - s) = -\left\langle X(t) \left. \frac{\partial \ln \rho(X,t)}{\partial X} \right| t=s \right\rangle
  \]
  $\rho(X,t)$: probability distribution function of $X$.
  **If the distribution $\rho$ is not Gaussian, $G \not\propto C$ in general!**
Typical examples

- Correlation function $C(t - s) = \langle X(t)X(s) \rangle$
- Response function $G(t - s) = \left\langle \frac{\delta X(t)}{\delta f(s)} \right\rangle$

Gaussian system $G \propto C$

non-Gaussian system $G \not\propto C$
Implication to statistical theory of turbulence

- Incompressible Navier-Stokes eq. with forcing

\[
\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0. \tag{1}
\]

- Expression in the Fourier space \([\mathbf{u}(x, t) = \sum_k \hat{\mathbf{u}}(k, t)e^{ik \cdot x}]

\[
\partial_t \hat{u}_j(k, t) = \frac{-1}{2} \sum_{l,m=1}^3 P_{jlm}(k) \sum_{\frac{p+q}{k+p+q}=0} \hat{u}_l(-p, t) \hat{u}_m(-q, t) - \nu|k|^2 \hat{u}_j(k, t) + \hat{f}_j(k, t) \tag{2}
\]

Holy grail: closure eq. of the correlation func. \(C_{jl}(k, t, s) = \langle \hat{u}_j(k, t) \hat{u}_l(-k, s) \rangle\)

- Direct interaction approximation (DIA), (R.H. Kraichnan 1959)
  - Decompose \(\hat{u} \Rightarrow \hat{u} + \delta\hat{u}\) and get the linearized eq. of \(\delta\hat{u}\) from (2).
  - Response func. of the linearized eq.: \(G_{jl}(k, t, s) = \langle \frac{\delta\hat{u}_j(k, t)}{\delta\hat{u}_l(-k, s)} \rangle\)
  - Closure eqs. of \(C\) and \(G\) (under statistical homogeneity and isotropy):
    \[
    [\partial_t + \nu k^2 + F_1(C, k, t, t')]C(k, t, t') = 0,
    [\partial_t + \nu k^2 + F_1(C, k, t, t')]G(k, t, t') = 0,
    (\partial_t + 2\nu k^2)C(k, t, t') = \int dk' \int dk'' F_2(k, k', k'') \int_{-\infty}^t ds C(k', t, t') [G(k', t') C(k'', t, t') - G(k', t, t') C(k, t, t')]\]
  - These closure eqs. are solved by (naturally) assuming \(G \propto C\).
• Our final goal is $G$ and $C$ of turbulence but....

• **Problem**: calculation of the response function $G_{jl}(k, t, s) = \left\langle \frac{\delta \hat{u}_j(k, t)}{\delta \hat{u}_l(-k, s)} \right\rangle$
  
  is numerically costly!

• Less-costly models of turbulence
  - Turbulence in two dimensions
  - Burgers equation (Burgers turbulence) $\partial_t u + (u \cdot \nabla)u = \nu \nabla^2 u$.
  - “shell models” of turbulence
  - …
    * shell models (reduced dynamical-system models)
      • homogeneous, isotropic turbulence (Obukhov 1971; Gledzer 1973; Yamada & Ohkitani 1987).
      • thermal convection turbulence (Suzuki & Toh 1995).
      • magnetohydrodynamic (MHD) turbulence (Hattori & Ishizawa 2001).
      • quantum turbulence (Wacks & Barenghi 2011).
      • …
Dynamical-system model of turbulence: shell model

- Gledzer-Ohkitani-Yamada shell model (Yamada & Ohkitani 1987)

\[
\left(\frac{d}{dt} + \nu k_j^2\right)u_j(t) = i[k_j u_{j+2}u_{j+1}^* - \frac{1}{2}k_{j-1} u_{j+1}u_{j-1}^* - \frac{1}{2}k_{j-2} u_{j-1}u_{j-2}] + f_j
\]

\[u_j(t) \in \mathbb{C}; \quad j = 1, \ldots, N; \quad k_j = k_0 2^j; \quad \ast \text{ complex conjugate.}\]

- “shell”: annulus in the wavenumber space \(2^j \leq |k| \leq 2^{j+1}\)

- This is a minimalistic model of the Navier-Stokes eq. in the Fourier space

\[
\mathbf{u}(x, t) = \sum_{\mathbf{k}} \hat{\mathbf{u}}(\mathbf{k}, t) e^{i\mathbf{k} \cdot x},
\]

\[
(\partial_t + \nu |\mathbf{k}|^2)\hat{u}_n(\mathbf{k}, t) = -\frac{1}{2} \sum_{l,m=1}^{3} P_{nlm}(\mathbf{k}) \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=\mathbf{0}} \hat{u}_l(-\mathbf{p}, t)\hat{u}_m(-\mathbf{q}, t) + \hat{f}_n(\mathbf{k}, t).
\]

\[u_j(t) \text{ of } k_j \text{ is a representative of } \hat{\mathbf{u}}(\mathbf{k}, t) \text{ in the } j\text{-th shell } k_j \leq |k| \leq k_{j+1}.\]

- This Gledzer-Ohkitani-Yamada shell model has a lot of success.
Success of the Gledzer-Ohkitani-Yamada shell model

\[
\left( \frac{d}{dt} + \nu k_j^2 \right) u_j(t) = i[k_j u_{j+2} u_{j+1}^* - \frac{1}{2} k_{j-1} u_{j+1} u_{j-1}^* - \frac{1}{2} k_{j-2} u_{j-1} u_{j-2}] + f_j
\]

\[(u_j(t) \in \mathbb{C}; \quad j = 1, \ldots, 24; \quad k_j = k_0 2^j; \quad f_j \text{ is zero except } f_1 = 0.5 + 0.5i).\]

The shell model can reproduce some characteristics of the Navier-Stokes turbulence.

Energy spectrum \( \frac{|u_j|^2}{k_j} \)  

\[|u_j|^2 \propto k_j^{-5/3} \]

\[\langle |u_j|^p \rangle \propto k_j^{-\zeta_p} \]

- Scaling exponents \( \zeta_p \) of the moments coincides with those of the NS turbulence

\[\left\langle \left\{ \left[ u(x + r) - u(r) \right] \cdot \frac{r}{|r|} \right\}^p \right\rangle \propto |r|^{\zeta_p}. \]
Correlation and response functions of the shell model

\[
\left( \frac{d}{dt} + \nu k_j^2 \right) u_j(t) = i[k_j u_{j+2} u_{j+1}^* - \frac{1}{2} k_{j-1} u_{j+1} u_{j-1}^* - \frac{1}{2} k_{j-2} u_{j-1} u_{j-2}] + f_j
\]

\[(u_j(t) \in \mathbb{C}; \quad j = 1, \ldots, 24; \quad k_j = k_0 2^j; \quad f_j \text{ is zero except } f_1 = 0.5 + 0.5i)\]

- Correlation and response functions in the inertial range

\[|u_j|^2 \sim k^{-5/3}\]

**Response function** \[G(t - s) = \text{Re} \left\langle \frac{\delta u_{11}(t)}{\delta u_{11}(s)} \right\rangle \quad (\delta u_{11}(s) \text{ is purely real})\]

**Auto correlation function** \[C(t - s) = \frac{\langle \text{Re}[u_{11}(t)] \text{ Re}[u_{11}(s)] \rangle}{\langle \{\text{Re}[u_{11}(t)]\}^2 \rangle} \quad \text{(normalized)}\]
• The fluctuation-dissipation relation $G \propto C$ breaks down for the shell model (Biferale et al. 2002).

![Graph showing C(t−s), G(t−s)]

• An expression for this discrepancy between $G$ and $C$ for the shell model?

After trial and error, we find:

A relation between $G$ and $C$ can be obtained if we add noises to the shell model

$$\frac{d}{dt} u_j = i[k_j u_{j+2} u_{j+1}^* - \frac{1}{2} k_{j-1} u_{j+1} u_{j-1}^* - \frac{1}{2} k_{j-2} u_{j-1} u_{j-2}] + f_j - \nu k_j^2 u_j + \xi_j.$$ 

$\xi_j$: Gaussian white noise: $\langle \xi_j(t) \rangle = 0$, $\langle \xi_j(t) \xi_{\ell}^*(s) \rangle = 2 \nu k_j^2 T \delta_{j,\ell} \delta(t-s)$. 
Property of the shell model with noise

\[ \frac{d}{dt} u_j = i[k_j u_{j+2} u^*_{j+1} - \frac{1}{2} k_{j-1} u_{j+1} u^*_{j-1} - \frac{1}{2} k_{j-2} u_{j-1} u_{j-2}] - \nu k^2_j u_j + f_j + \xi_j, \]

\[ \langle \xi_j(t) \xi^*_\ell(s) \rangle = 2\nu k^2_j T \delta_{j,\ell} \delta(t-s). \]

If “the heat bath temperature” \( T \) is small enough, the noisy model is close to the Navier-Stokes turbulence.
The noisy shell model: correlation and response functions

\[ \frac{d}{dt} u_j = i[k_j u_j u_{j+1}^* - \frac{1}{2} k_{j-1} u_{j+1} u_{j-1}^* - \frac{1}{2} k_{j-2} u_{j-1} u_{j-2}] - \nu k_j^2 u_j + f_j + \xi_j, \]
\[ \langle \xi_j(t) \xi_j^*(s) \rangle = 2\nu k_j^2 T \delta_{j,\ell} \delta(t-s). \]
Noisy shell model: Correlation function $C$ and response function $G$

$$\frac{d}{dt} u_j = i[k_j u_{j+2} u_{j+1}^* - \frac{1}{2} k_{j-1} u_{j+1} u_{j-1}^* - \frac{1}{2} k_{j-2} u_{j-1} u_{j-2}] - \nu k_j^2 u_j + f_j + \xi_j,$$

$$\langle \xi_j(t) \xi^*_\ell(s) \rangle = 2\nu k^2_j T \delta_{j,\ell} \delta(t-s).$$

$$G_{jj}(t) = \frac{\delta u_j(t)}{\delta u_j(0)}, \quad C_{jj}(t) = \langle u_j(t) u_j^*(0) \rangle,$$

$$H_{jj}(t) = \frac{1}{T} C_{jj}(t) - \frac{1}{2\nu k^2_j T} \left[ \langle u_j(t) \Lambda_j^*(0) \rangle + \langle u_j(0) \Lambda_j^*(t) \rangle \right].$$

$$(\Lambda_j(t) = i[k_j u_{j+2} u_{j+1}^* - \frac{1}{2} k_{j-1} u_{j+1} u_{j-1}^* - \frac{1}{2} k_{j-2} u_{j-1} u_{j-2}] + f_j).$$
Noisy shell model: correlation function $C$ and response function $G$

$$\frac{d}{dt} u_j = \text{i}[k_j u_{j+2}u_{j+1}^* - \frac{1}{2}k_{j-1}u_{j+1}u_{j-1}^* - \frac{1}{2}k_{j-2}u_{j-1}u_{j-2}] + f_j - \nu k_j^2 u_j + \xi_j,$$

$$\langle \xi_j(t)\xi_{\ell}^*(s) \rangle = 2\nu k_j^2 T \delta_{j,\ell} \delta(t-s).$$

- The response function is expressed with triple correlations

$$G_{j,j}(t-s) \overset{\text{def}}{=} \frac{\delta u_j(t)}{\delta u_j(s)} = \frac{1}{T} C_{j,j}(t-s) - \frac{1}{2\nu k_j^2 T} \left[ \langle u_j(t)\Lambda_j^*(s) \rangle + \langle u_j(s)\Lambda_j^*(t) \rangle \right].$$

- Derivation (Harada & Sasa 2006):

$$\langle u_j(t) \rangle = \int du_0 \int [du] \rho_0(u_0) T[u|u_0(t_0)] u_j$$

with the assumption that the transition probability $T[u|u_0(t_0)]$ is determined by the noise

$$[du]T[u|u_0(t_0)] \propto [d\xi] \exp \left[ -\frac{1}{2} \sum_{j=1}^{N} \int_{t_0}^{t} ds \frac{\xi_j(s)^2}{\sigma_j^2} \right]$$

$$\sigma_j = \nu k_j^2 T.$$
Summary and outlook

- In non-Gaussian systems, the correlation $C$ and response functions $G$: $C \not\propto G$.

- What about fluid turbulence? Expression of the discrepancy?

- The Gledzer-Ohkitani-Yamada shell model: a dynamical-system model of turbulence

- In the shell model, $C \not\propto G$ as expected.

- The shell model with noise: again $C \not\propto G$

\[
\partial_t u_j = i [k_j u_{j+2} u_{j+1}^* - \frac{1}{2} k_{j-1} u_{j+1} u_{j-1}^* - \frac{1}{2} k_{j-2} u_{j-1} u_{j-2}^*] + f_j - \nu k_j^2 u_j + \xi_j,
\]

\[
\langle \xi_j(t) \xi_{\ell}^*(s) \rangle = 2\nu k_j^2 T \delta_{j,\ell} \delta(t-s), \quad (k_j = k_0 2^j, \ u_j \in \mathbb{C}).
\]

- For the noisy shell model, the expression between $G$ and $C$ ($C_{jj}(t) = \langle u_j(t) u_j^*(0) \rangle$)

\[
G_{jj}(t-s) = \frac{\delta u_j(t)}{\delta u_j(s)} = \frac{1}{T} C_{jj}(t-s) - \frac{1}{2\nu k_j^2 T} \left[ \langle u_j(t) \Lambda_j^*(s) \rangle + \langle u_j(s) \Lambda_j^*(t) \rangle \right].
\]

- $\langle u_j(t) \Lambda_j^*(s) \rangle$ resembles the energy transfer among the shell.

- How about the (noisy?) Navier-Stokes turbulence???