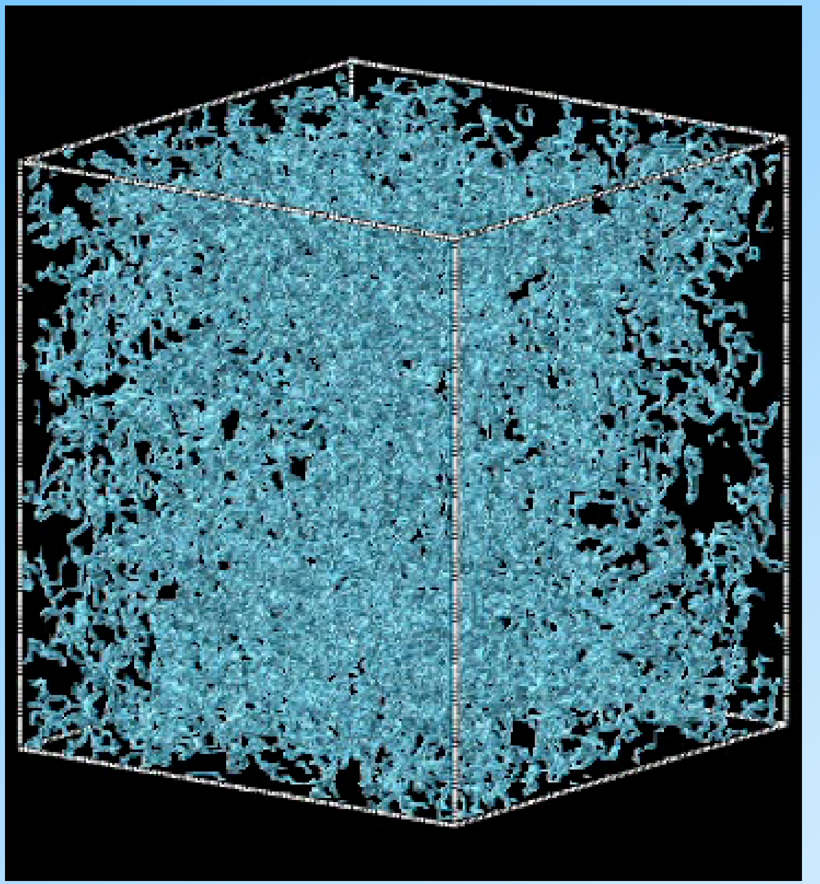
Quantized Vortices and Quantum Turbulence



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Review article

- M. Tsubota, J. Phys. Soc. Jpn.77 (2008) 111006
- Progress in Low Temperature Physics Vol.16, eds. W. P. Halperin and M. Tsubota, Elsevier, 2009

What is “quantum” ?

Element of something

What is “quantum mechanics” ?

Mechanics with element

Energy, momentum and angular momentum *etc*. are quantized.

The element is determined by the Planck’s constant *h*.

What is “quantum turbulence” ?

Turbulence with some “element”

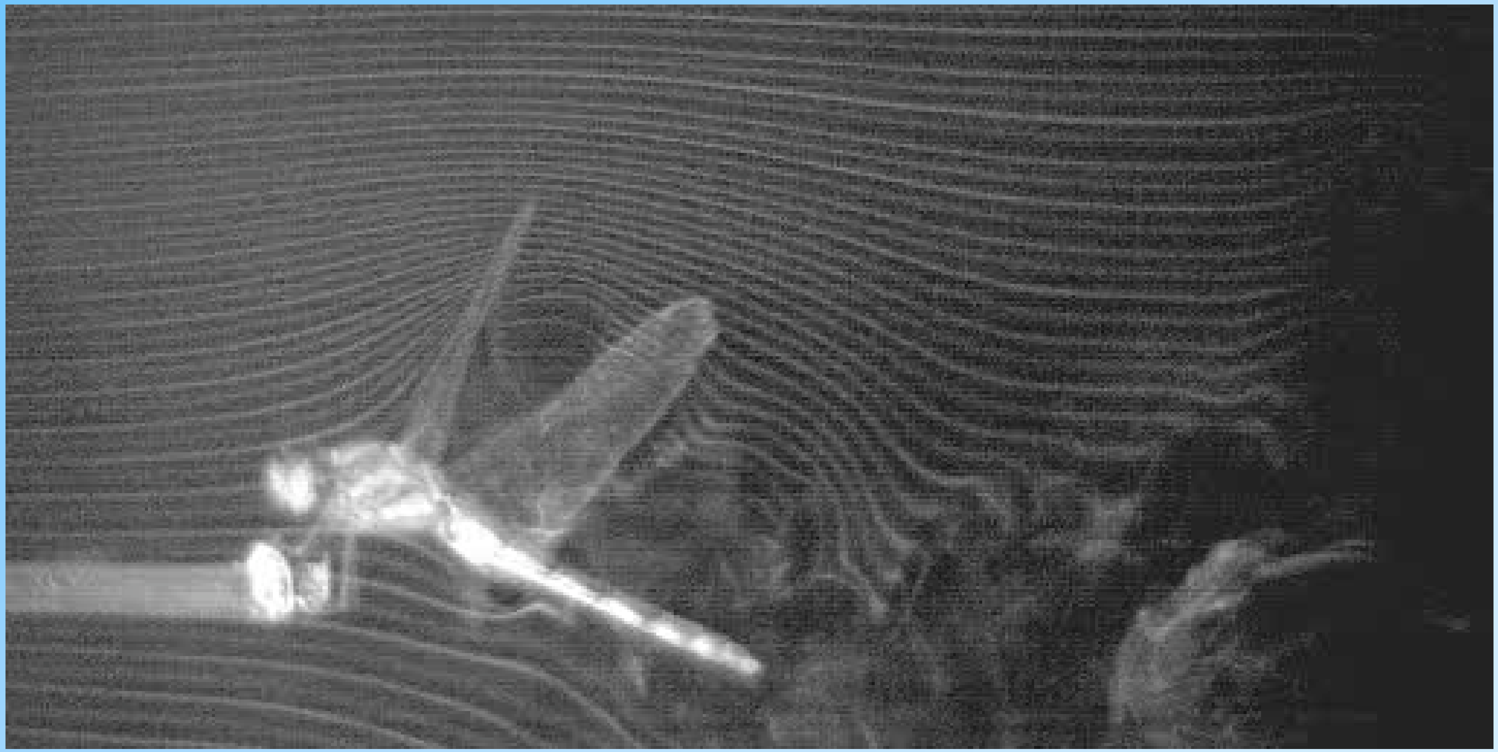
Leonardo Da Vinci
(1452-1519)

Da Vinci observed turbulent flow and found that turbulence consists of many vortices with different scales.

Turbulence is not a simple disordered state but having some structures with vortices.

Certainly turbulence looks to have many vortices.

Turbulence behind a dragonfly

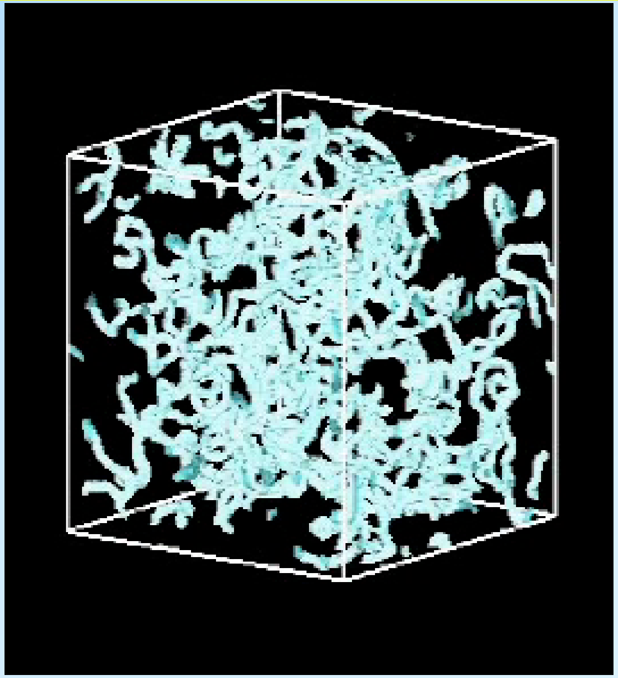


http://www.nagare.or.jp/mm/2004/gallery/iida/dragonfly.html

It is not so straightforward to confirm the Da Vinci message in classical turbulence.

Key concept

The Da Vinci message "*turbulence consists of vortices*" is actually realized in quantum turbulence (QT) comprised of quantized vortices.



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   Basics of Quantum Hydrodynamics of the GP(Gross-Pitaevskii) model, Brief research history of QT

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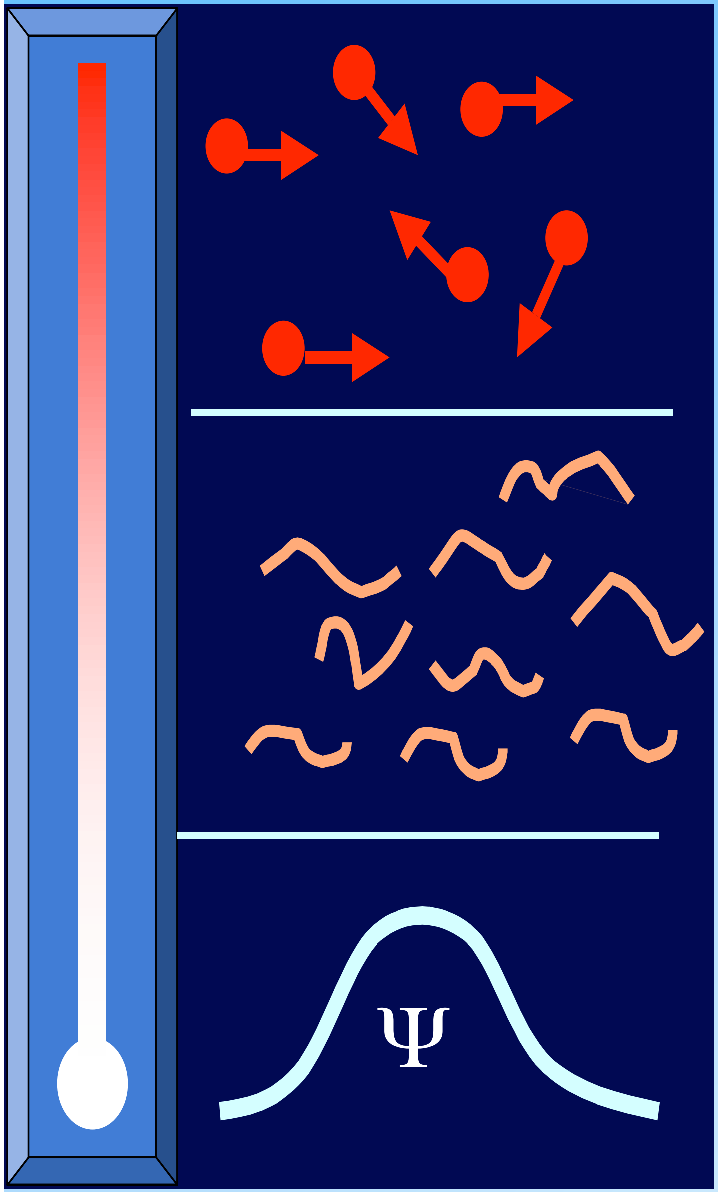
2. QT by the GP model -Energy spectrum-
3. QT in atomic BECs
4. Quantized vortices in two-component BECs

   Quantum Kelvin-Helmholtz instability, QT

0. Introduction



Quantum mechanics

～ Duality of matter and wave ～

Each atom behaves as a particle at high temperatures.

Thermal de Broglie wave length
～ Distance between particles

Each atom behaves like a wave at low temperatures.

Bose-Einstein condensation (BEC)

Each atom occupies the same single particle ground state. The matter waves become coherent, making a macroscopic wave function Ψ .

Basics of quantum hydrodynamics of the GP model (1)

The wave function Ψ obeys the Gross-Pitaevskii (GP) equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\left(\frac{\hbar^2}{2m}\nabla^2 + \mu\right)\Psi + g|\Psi|^2\Psi \qquad (1)$$

When we use the expression $\Psi(\mathbf{r},t) = \sqrt{n_0(\mathbf{r},t)}\exp[i\theta(\mathbf{r},t)]$, the real and imaginary parts of Eq. (1) are reduced to

$$\frac{\partial\sqrt{n_0}}{\partial t} = -\frac{\hbar}{2m}\left(2\nabla\sqrt{n_0}\nabla\theta + \sqrt{n_0}\nabla^2\theta\right) \qquad (2)$$

$$\hbar\frac{\partial\theta}{\partial t} = -\frac{\hbar^2}{2m}\left\{(\nabla\theta)^2 - \frac{\nabla^2\sqrt{n_0}}{\sqrt{n_0}}\right\} + \mu - gn_0 \qquad (3)$$

Basics of quantum hydrodynamics of the GP model (2)

$$\frac{\partial\sqrt{n_0}}{\partial t} = -\frac{\hbar}{2m}\left(2\nabla\sqrt{n_0}\nabla\theta + \sqrt{n_0}\nabla^2\theta\right) \qquad (2)$$

Equation (2) is a continuity equation of the condensate.

The flux density $\mathbf{j} = -\frac{i\hbar}{2m}\left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right)$ with $\Psi = \sqrt{n_0}\exp[i\theta]$ gives

$$\mathbf{j} = n_0\mathbf{v}_s, \quad \mathbf{v}_s \equiv \frac{\hbar}{m}\nabla\theta$$

$$\frac{\partial n_0}{\partial t} = -\nabla\mathbf{j}$$

Superflow $\mathbf{v}_s$ is driven by the potential θ which is the phase of the wave function.

Basics of quantum hydrodynamics of the GP model (3)

$$\hbar \frac{\partial \theta}{\partial t} = -\frac{\hbar^2}{2m}\left\{ (\nabla\theta)^2 - \frac{\nabla^2 \sqrt{n_0}}{\sqrt{n_0}} \right\} + \mu - g n_0 \qquad (3)$$

Equation (3) with $v_s = \frac{\hbar}{m}\nabla\theta$ leads to the equation of superflow

$$\frac{\partial \mathbf{v}_s}{\partial t} + (\mathbf{v}_s \cdot \nabla)\mathbf{v}_s = \frac{1}{m}\nabla\left(\mu - g n_0 + \frac{\hbar}{2m}\frac{\nabla^2\sqrt{n_0}}{\sqrt{n_0}} \right) \qquad (4)$$

Equation (4) is quite similar to the Euler equation of a perfect fluid, but has a different term of "quantum pressure".

The quantum pressure plays an important role in nucleation and reconnection of quantized vortices.

Summary of this part

GP Eq. $$i\hbar\frac{\partial\Psi}{\partial t} = -\left(\frac{\hbar^2}{2m}\nabla^2 + \mu\right)\Psi + g|\Psi|^2\Psi$$ with $$\Psi = \sqrt{n_0}\exp[i\theta]$$

Continuity Eq. of the density n_0 $$\frac{\partial n_0}{\partial t} = -\nabla \mathbf{j}, \quad \mathbf{j} = n_0 \mathbf{v}_s$$

Euler-like Eq. of Superflow $$\frac{\partial \mathbf{v}_s}{\partial t} + (\mathbf{v}_s \cdot \nabla)\mathbf{v}_s = \frac{1}{m}\nabla\left(\mu - g n_0 + \frac{\hbar}{2m}\frac{\nabla^2\sqrt{n_0}}{\sqrt{n_0}}\right)$$

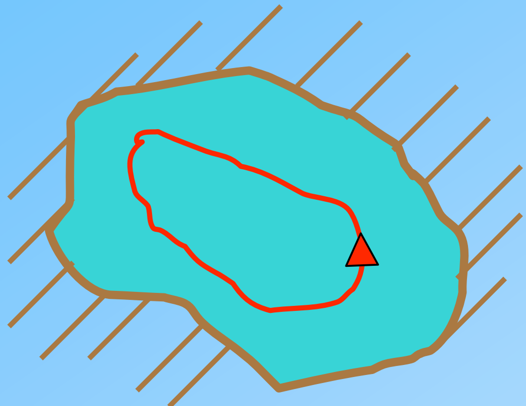
Superflow $$\mathbf{v}_s \equiv \frac{\hbar}{m}\nabla\theta$$

Basics of quantum hydrodynamics of the GP model (4)

Quantization of circulation

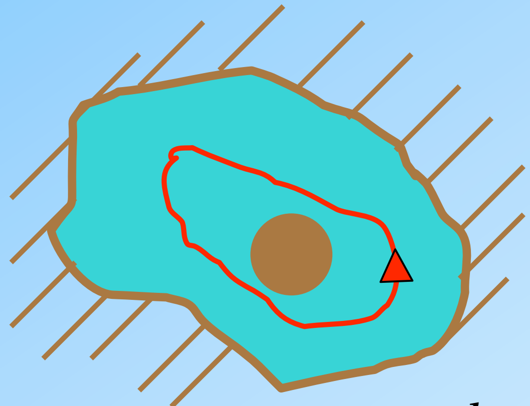
Superfow $\mathbf{v}_s = \dfrac{\hbar}{m}\nabla\theta$

Single-connected region



$$\Gamma = \oint_C \mathbf{v}_s \cdot d\mathbf{l} = 0, \ \ \mathrm{rot}\,\mathbf{v}_s = 0$$

Multi-connected region



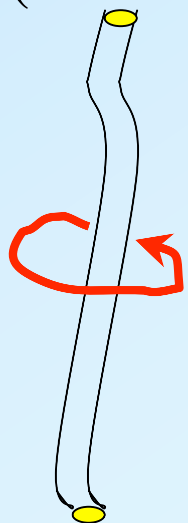
$$\Gamma = \oint_C \mathbf{v}_s \cdot d\mathbf{l} = \frac{\hbar}{m}\oint_C \nabla\theta \cdot d\mathbf{l} = \frac{h}{m}n \quad (n\text{: integer})$$

Quantized circulation

$$\kappa = \frac{h}{m}$$

A vortex with quantized circulation and vacant core

Quantized vortex



A quantized vortex is a vortex of superflow in a BEC. Any rotational motion in superfluid is sustained by quantized vortices.

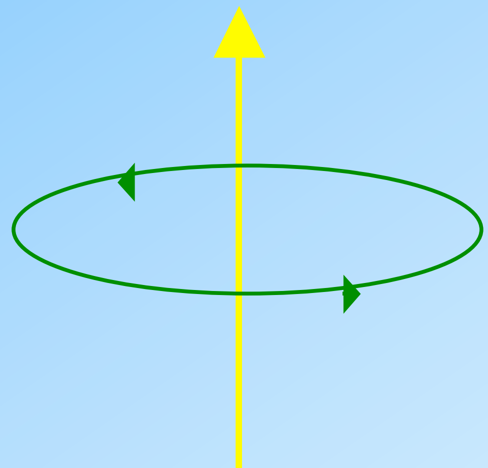
(i) The circulation is quantized.

$$\oint \mathbf{v}_s \cdot d\mathbf{s} = \kappa n \qquad (n = 0, 1, 2, \cdots)$$

$$\kappa = h / m$$

A vortex with $n \geqq 2$ is unstable.

⇨ **Every vortex has the same circulation.**

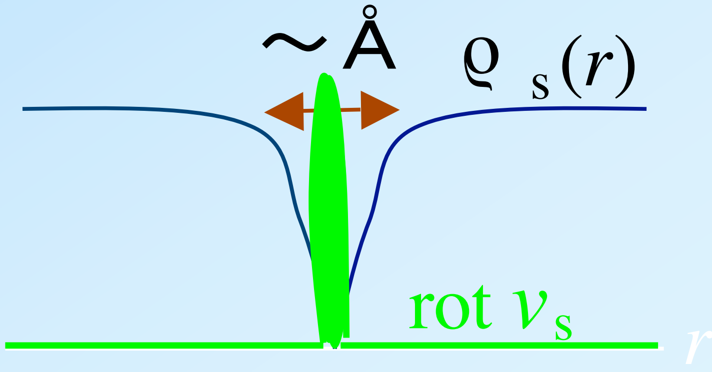


(ii) Free from the decay mechanism of the viscous diffusion of the vorticity.

⇨ **The vortex is stable.**

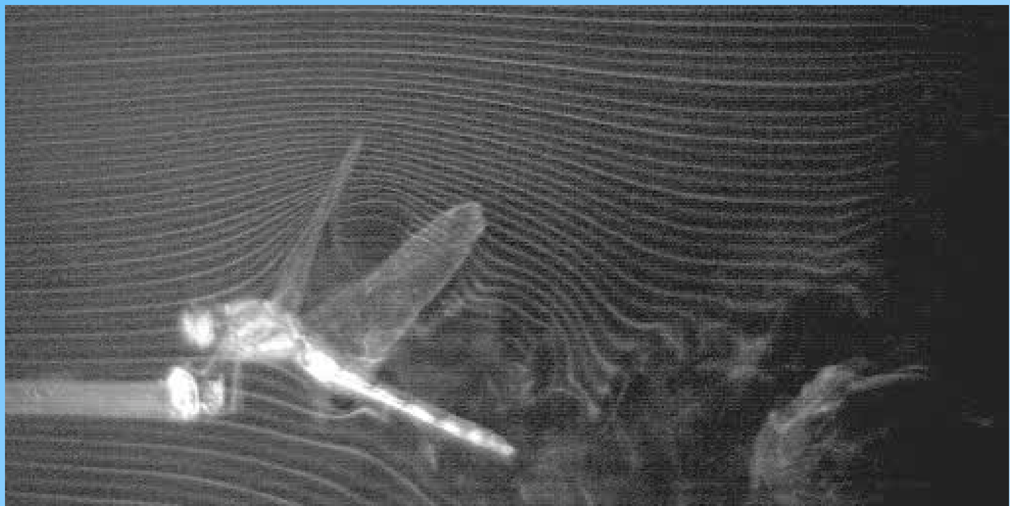
(iii) The core size is very small.

⇨ **The order of the coherence length.**

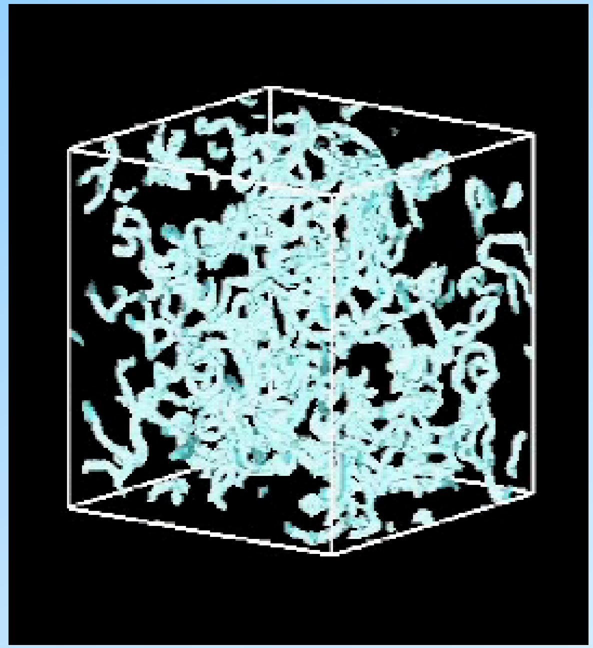


Classical Turbulence (CT) vs. Quantum Turbulence (QT)

Classical turbulence



Quantum turbulence



Motion of vortex cores

QT can be much simpler than CT, because each element of turbulence is well-defined.

・The quantized vortices are stable topological defects.
・Every vortex has the same circulation.
・Circulation is conserved.

Models available for simulation of QT

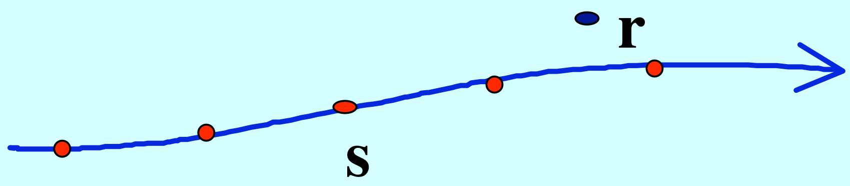
Gross-Pitaevskii (GP) model for the macroscopic wave function

$$\Psi(r) = \sqrt{n_0(r)}e^{i\theta(r)}$$

$$i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t} = \left[-\frac{\hbar^2\nabla^2}{2m} + V_{\text{ext}}(\mathbf{r}) + g|\Psi(\mathbf{r},t)|^2\right]\Psi(\mathbf{r},t)$$

Vortex filament model **Biot-Savart law**

$$\mathbf{v}_s(\mathbf{r}) = \frac{\kappa}{4\pi}\int\frac{(\mathbf{s}-\mathbf{r})\times d\mathbf{s}}{|\mathbf{s}-\mathbf{r}|^3}$$



A vortex makes the superflow of the Biot-Savart law, and moves with this local flow.

- Brief Research History of QT -

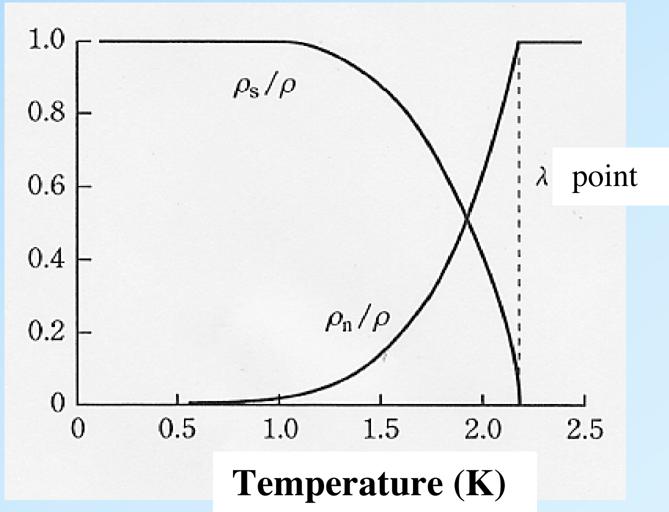
Liquid ^{4}He enters the superfluid state below 2.17 K (λ point) with Bose-Einstein condensation.

Its hydrodynamics are well described by the two-fluid model:

The two-fluid model (Tisza, Landau)

The system is a mixture of inviscid superfluid and viscous normal fluid.

$$\rho = \rho_s + \rho_n \qquad \mathbf{j} = \rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n$$



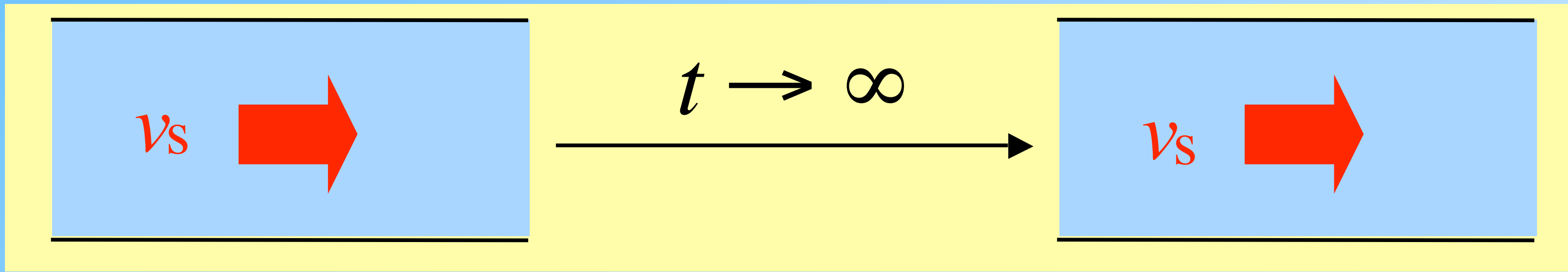
	Density	Velocity	Viscosity	Entropy
Superfluid	$\rho_s(T)$	$\mathbf{v}_s(\mathbf{r})$	0	0
Normal fluid	$\rho_n(T)$	$\mathbf{v}_n(\mathbf{r})$	$\eta_n(T)$	$s_n(T)$

The two-fluid model can explain various experimentally observed phenomena of superfluidity (e.g., the thermomechanical effect, film flow, etc.)

However, …

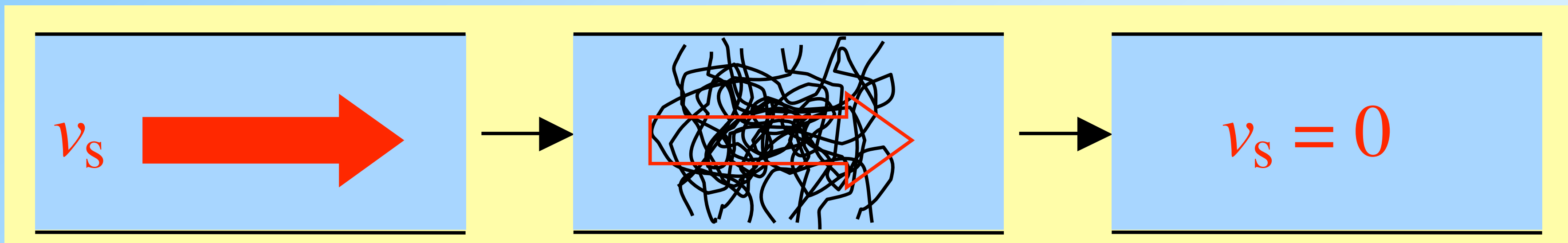
Superfluidity breaks down in fast flow

(i) $v_s < v_c$ (some critical velocity)



The two fluids do not interact so that the superfluid can flow forever without decaying.

(ii) $v_s > v_c$



A tangle of quantized vortices develops. The two fluids interact through mutual friction generated by tangling, and the superflow decays.

1955: **R. P. Feynman** proposed that "superfluid turbulence" consists of a tangle of quantized vortices.

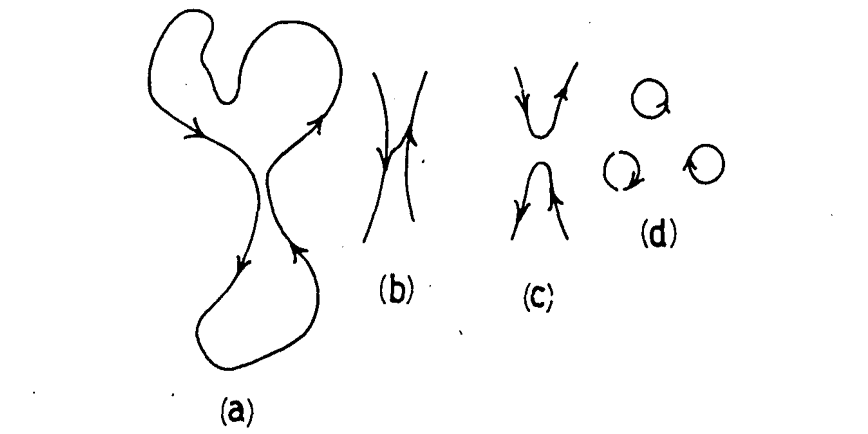
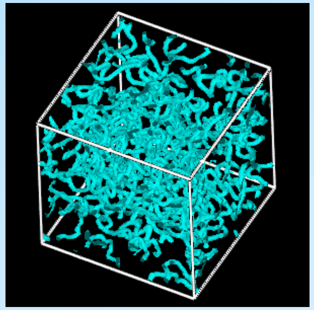


Fig. 10. A vortex ring (a) can break up into smaller rings if the transition between states (b) and (c) is allowed when the separation of vortex lines becomes of atomic dimensions. The eventual small rings (d) may be identical to rotons.

Progress in Low Temperature Physics Vol. I (1955), p.17

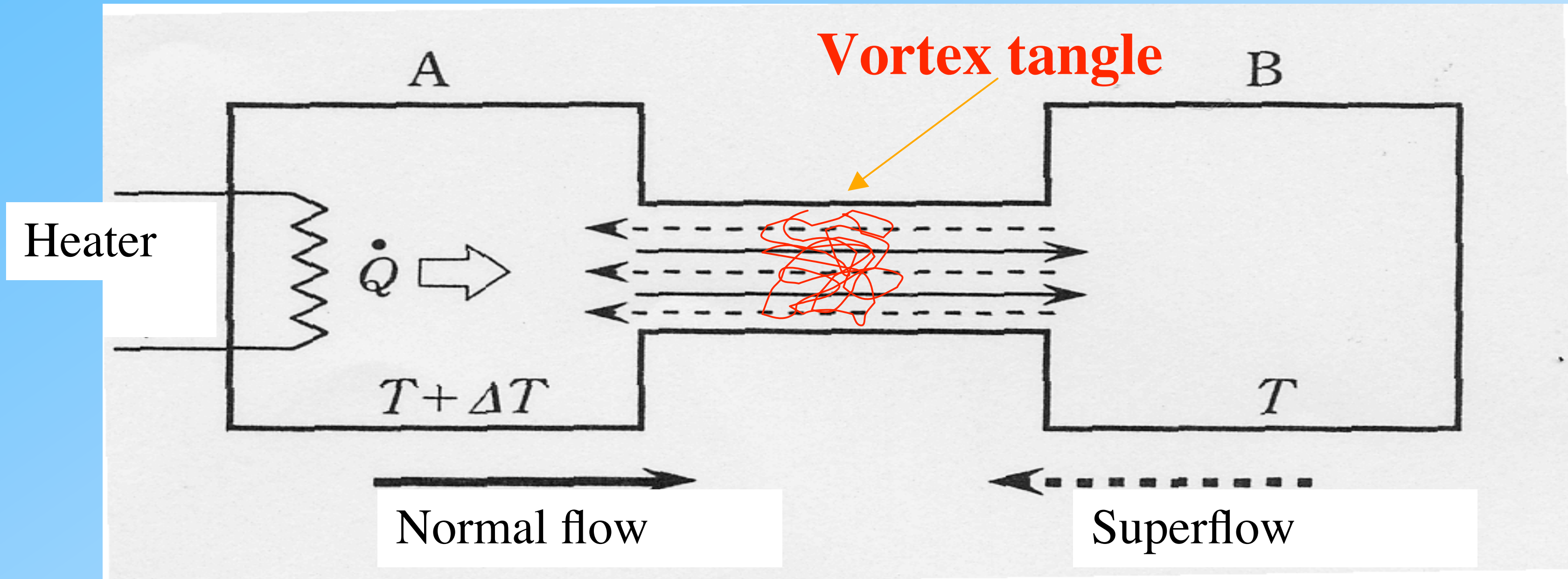
Such a large vortex should break up into smaller vortices like the cascade process in classical turbulence.



1955 – 1957: **W. F. Vinen** observed "superfluid turbulence".

Mutual friction between the vortex tangle and the normal fluid causes dissipation of the flow.

Many experimental studies were conducted chiefly on thermal counterflow of superfluid ^4He.

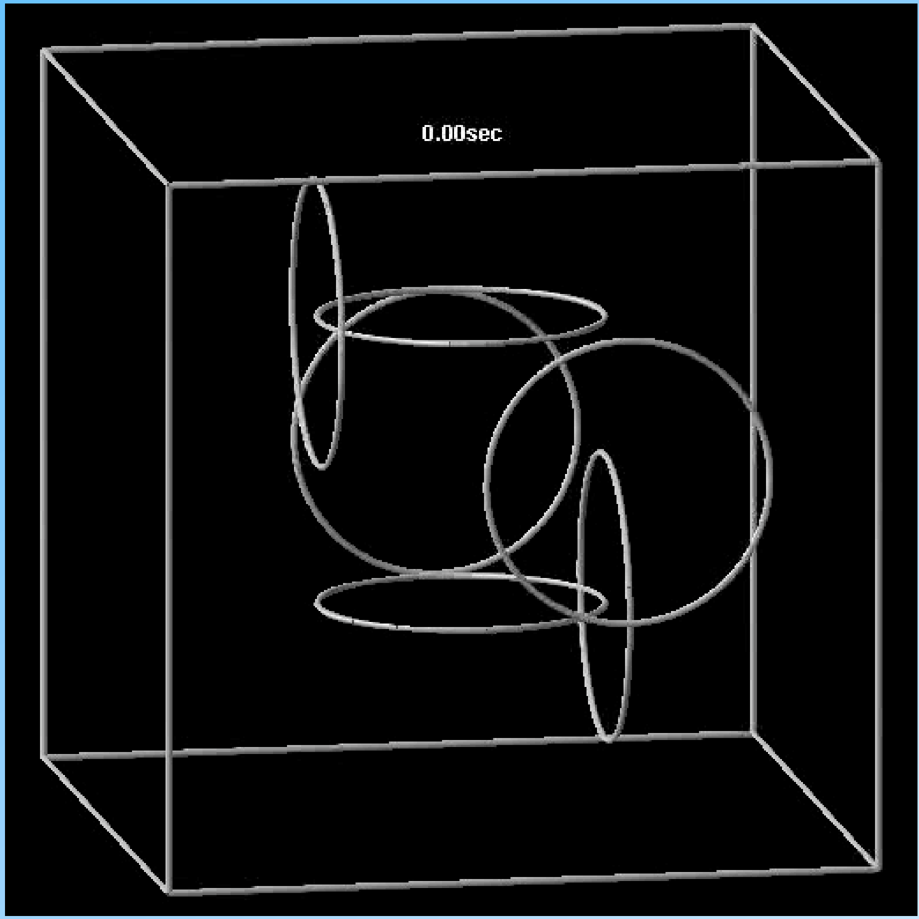


1980s **K. W. Schwarz** Phys. Rev. B38, 2398 (1988)

Performed a direct numerical simulation of the three-dimensional dynamics of quantized vortices and succeeded in quantitatively explaining the observed temperature difference *ΔT* .

Development of a vortex tangle in a thermal counterflow

Vortex filament model

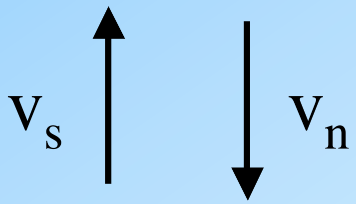


K. W. Schwarz, Phys. Rev. B38, 2398 (1988).

Schwarz obtained numerically the statistically steady state of a vortex tangle, which is sustained by the competition between the applied flow and the mutual friction.

H. Adachi, S. Fujiyama, MT, Phys. Rev. B81, 104511(2010)(**Editors suggestion**)

We made more correct simulation by taking the full account of the vortex interaction.



Counterflow turbulence has been successfully explained.

Most studies of superfluid tur
thermal counterflow.

⇨ **No analo**

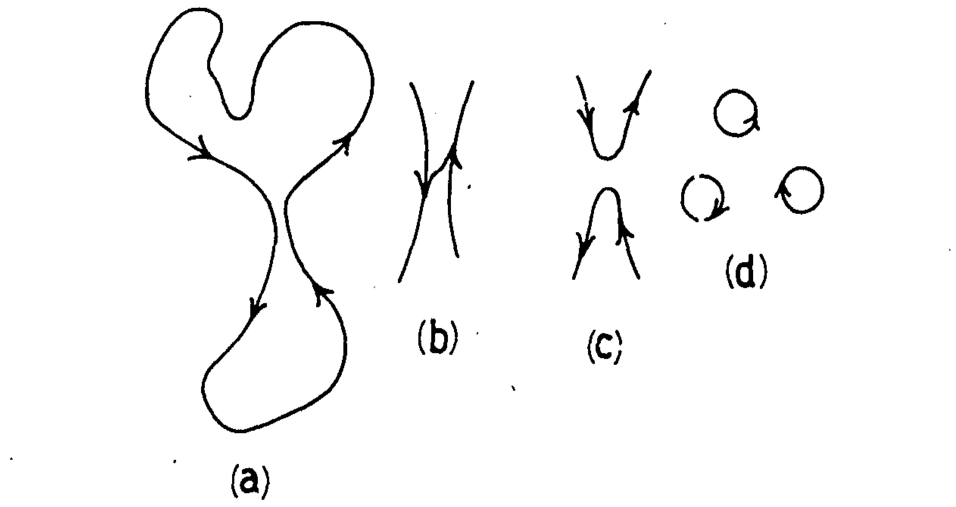


Fig. 10. A vortex ring (a) can break up into smaller rings if the transition between states (b) and (c) is allowed when the separation of vortex lines becomes of atomic dimensions. The eventual small rings (d) may be identical to rotons.

When Feynman drew the above figure, he was thinking of a cascade process in classical turbulence.

What is the relation between superfluid turbulence and classical turbulence ?

New era of quantum turbulence has come!

1. Superfluid helium

   Classical analogue has been considered since 1998.

   ～ *Energy spectrum of QT* ～

2. Atomic Bose-Einstein condensates (BECs)

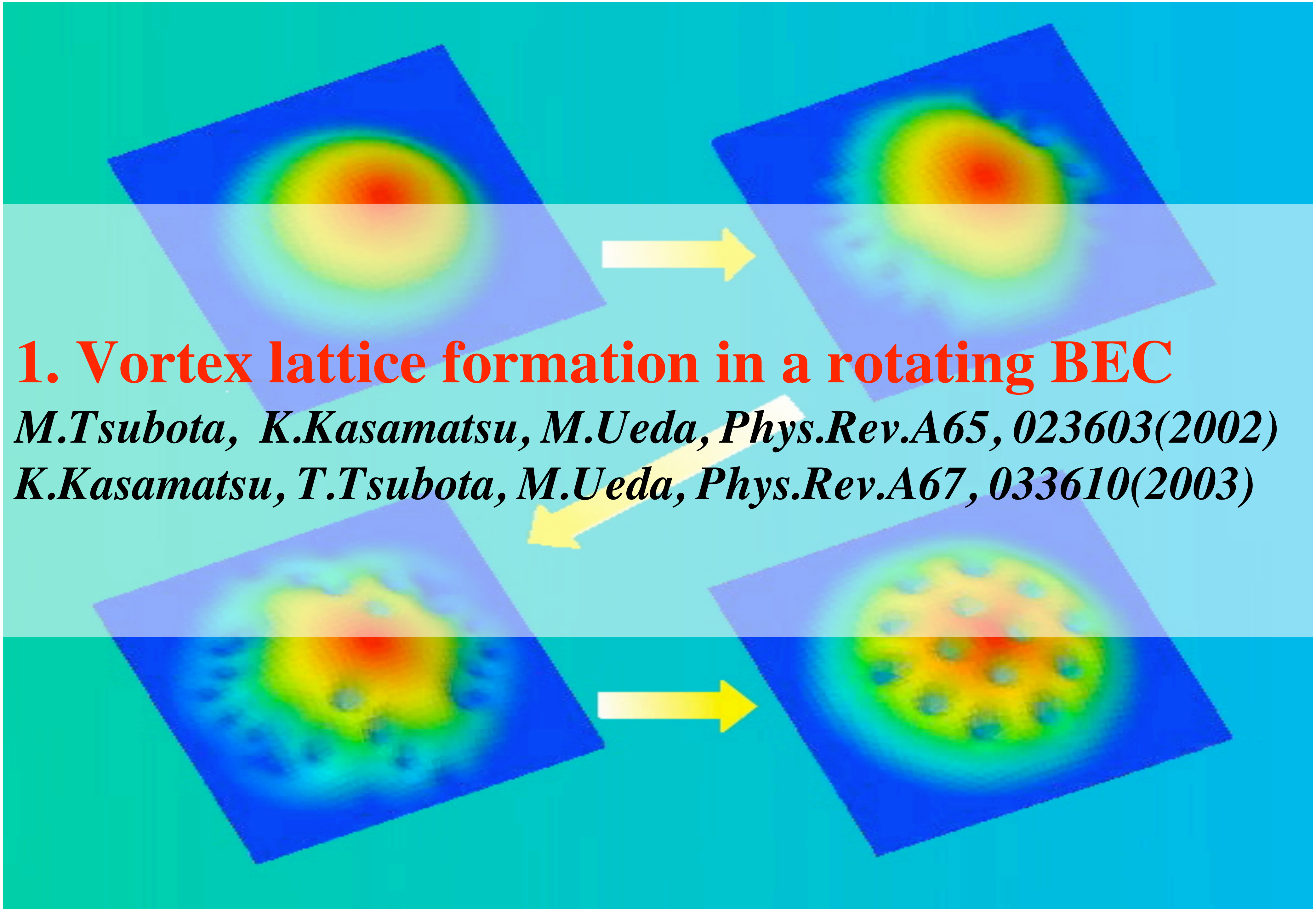
   BEC was realized in 1995.

Contents

1. Vortex lattice formation in a rotating BEC

M.Tsubota, K.Kasamatsu, M.Ueda, Phys.Rev.A65, 023603(2002)

K.Kasamatsu, T.Tsubota, M.Ueda, Phys.Rev.A67, 033610(2003)



Realization of atomic gas BEC

1995 ^{87}Rb, ^{7}Li, ^{23}Na

Laser cooling

An atom is subjected to a laser beam whose frequency is tuned to lie just below that of an atomic transition between an excited state and

Motion of atoms is reduced by laser.

the Doppler shift, reducing its velocity.

Cold atoms are collected at the focus of six laser beams. ***T*~ 100μK**

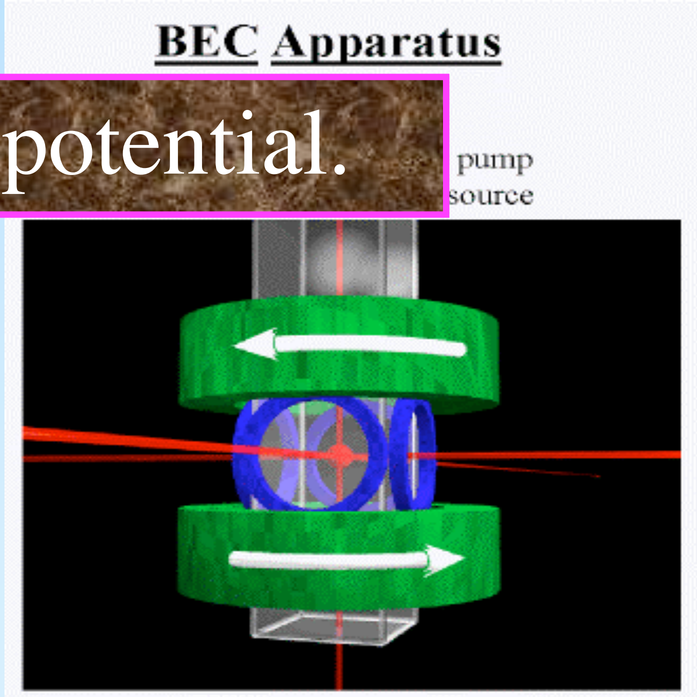
Magnetic trap

The atoms are trapped by potential.

Evaporation cooling

The fast atoms are released out of the trap.

BEC **T~ 100 nK**

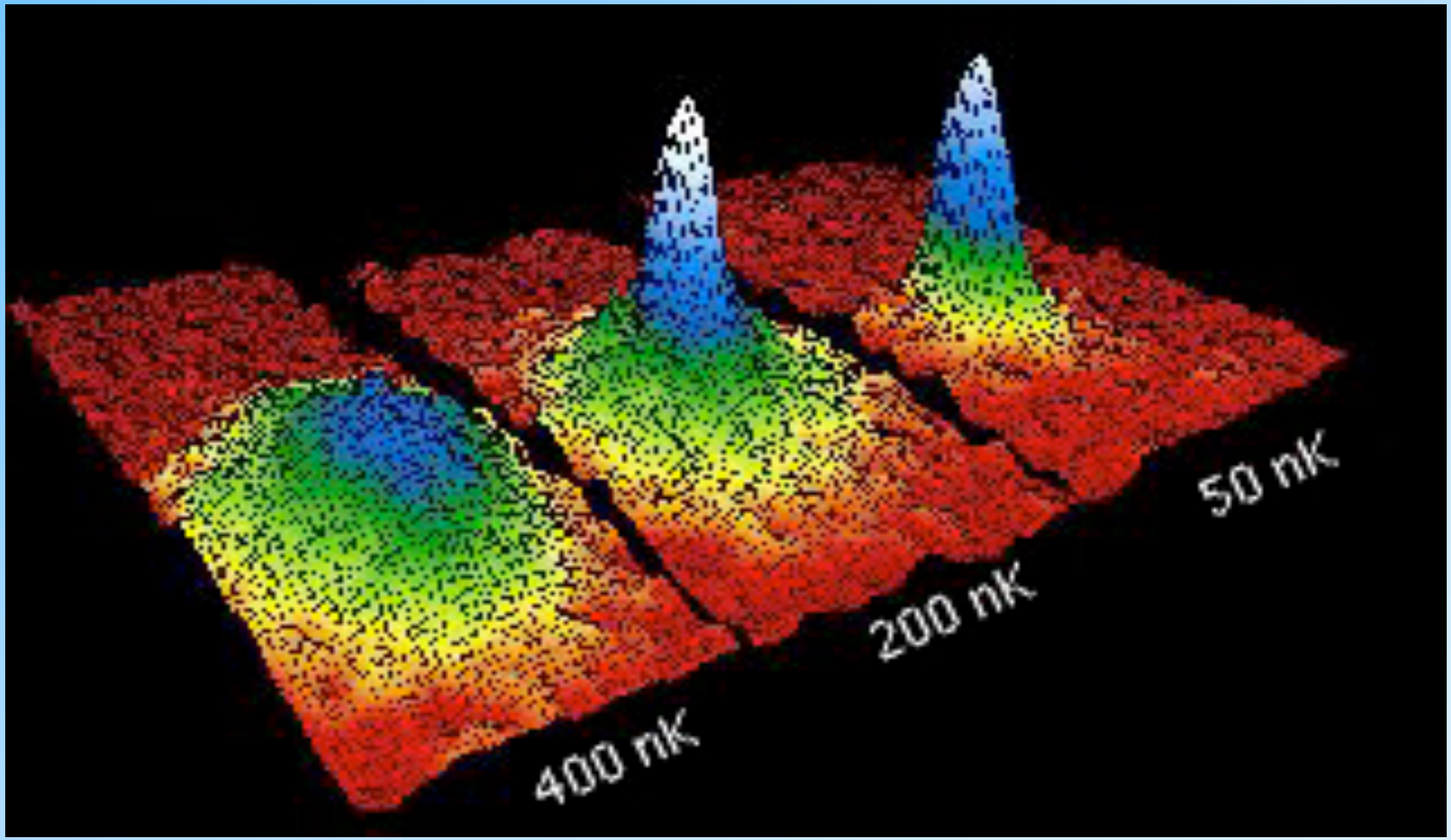


Observation of BEC

Turning off the trapping potential,

→ the gas expands with falling feely.

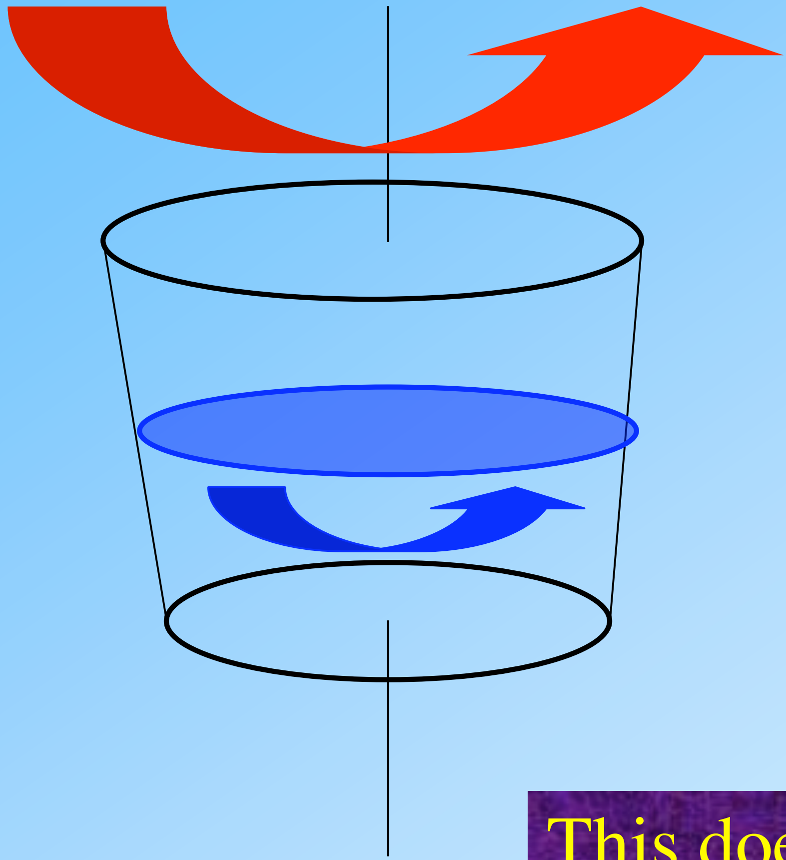
→ The observation of the position of atoms determines the initial distribution of velocity.



By MIT

The Nobel Prize in Physics 2001!

What happens if we rotate a vessel having a usual viscous classical fluid inside?



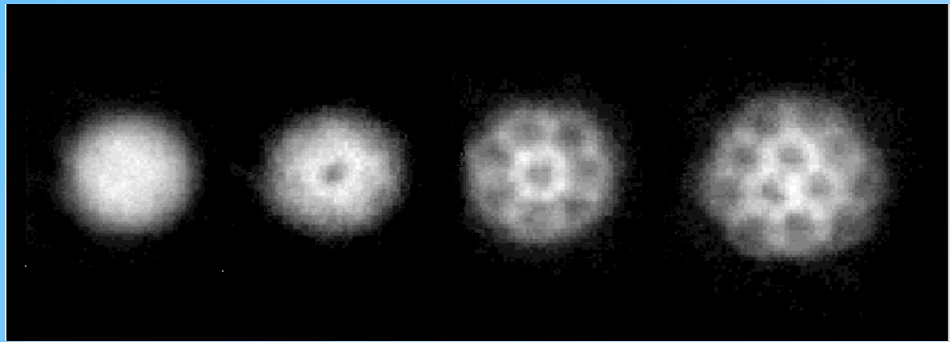
The fluid rotates with the same angular velocity with the vessel.

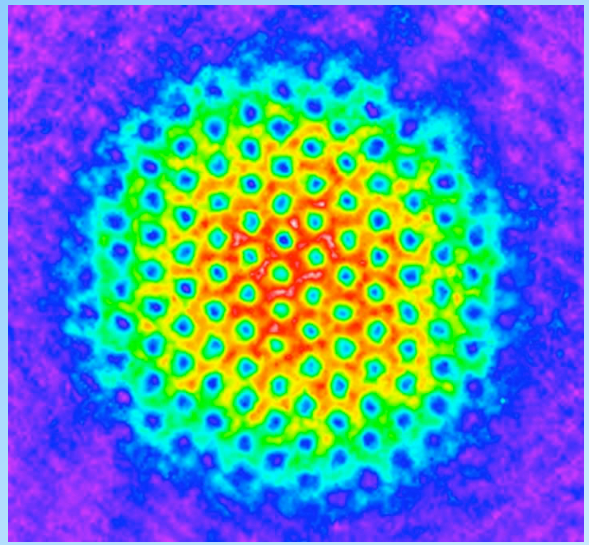
This means that there appears one vortex in the vessel. The single vortex can make the solid-body rotation with any angular velocity.

This does not occur in quantum fluids! Such experiments were done in atomic BECs.

Observation of quantized vortices in atomic BECs

ENS K.W.Madison, et al. PRL **84**, 806 (2000)

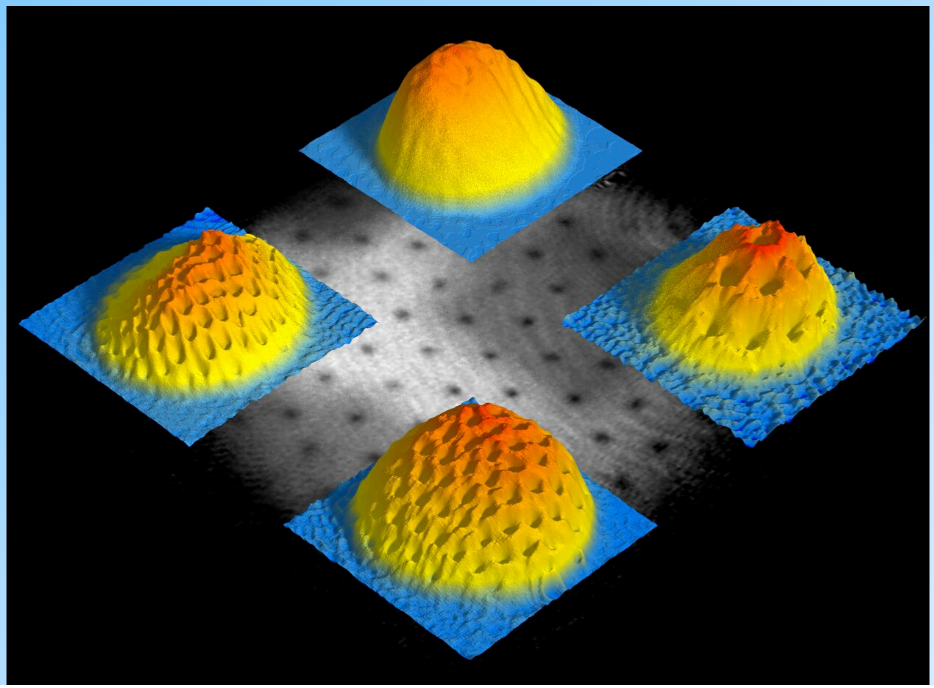




JILA

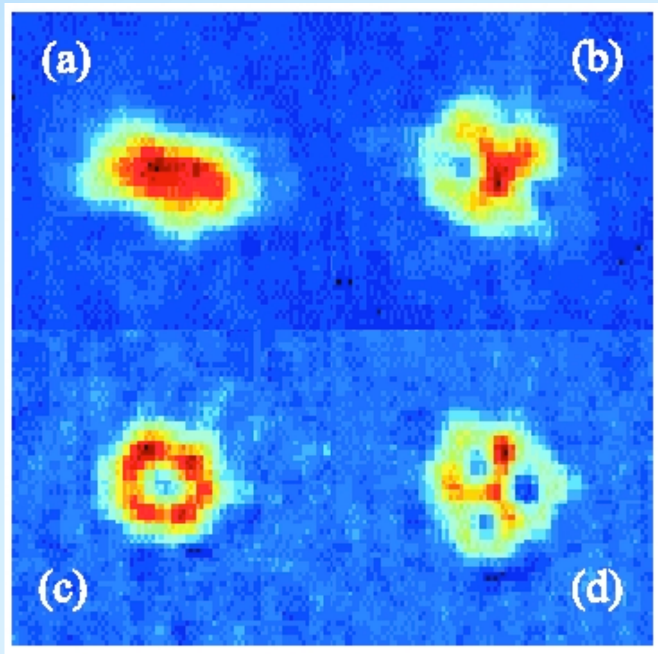
P. Engels, et al. PRL **87**, 210403 (2001)

MIT J.R. Abo-Shaeer, et al. Science **292**, 476 (2001)

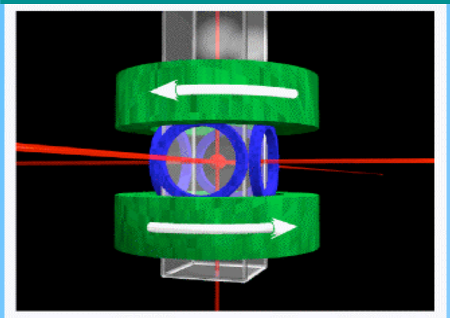


Oxford

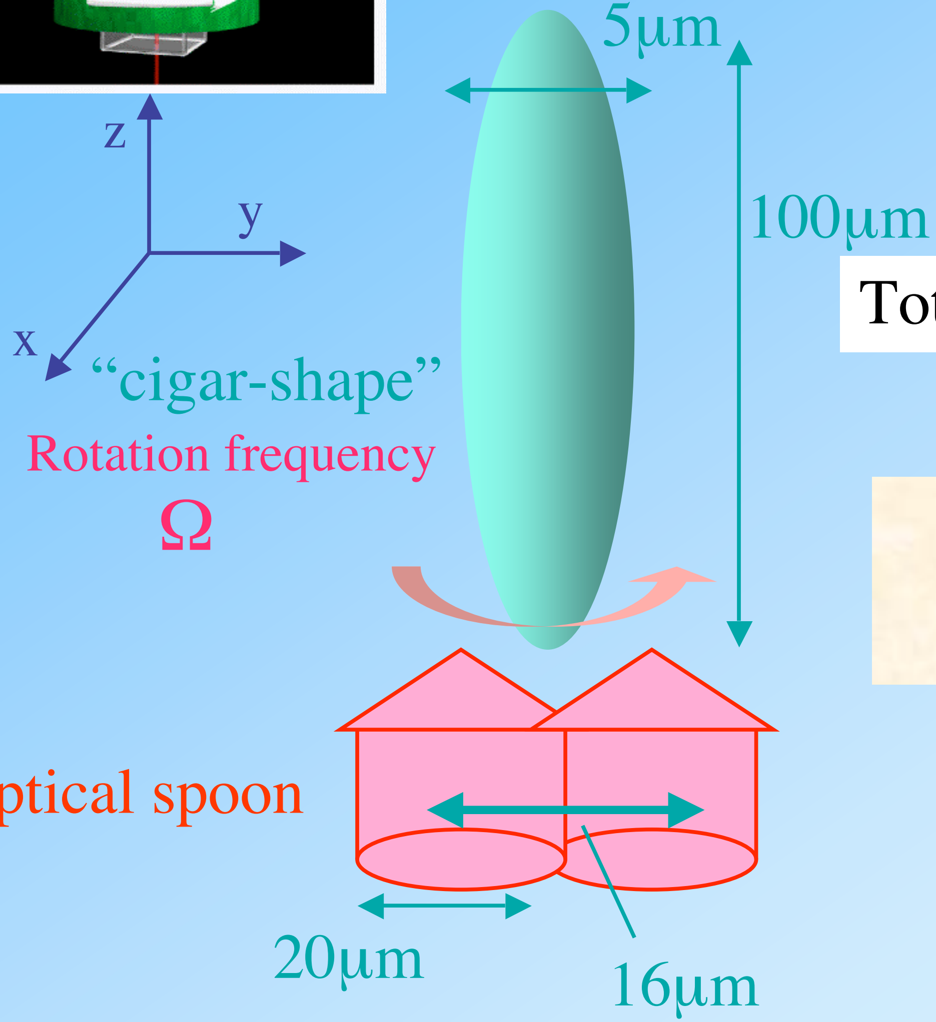
E. Hodby, et al. PRL **88**, 010405 (2002)



How can we rotate the trapped BEC？



K.W.Madison et al. Phys.Rev Lett **84**, 806 (2000)



Axisymmetric potential

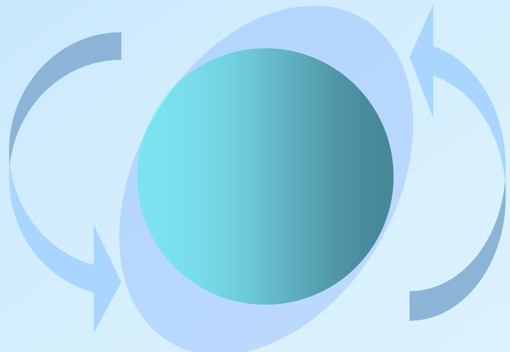
$$V_{ext}(\mathbf{R}) = V_{trap}(\mathbf{R}) + U_{stir}(\mathbf{R})$$

Total potential

Non-axisymmetric potential

$$U_{stir}(\mathbf{R}) = \frac{m}{2}\omega_{\perp}^{2}(\varepsilon_x X^2 + \varepsilon_y Y^2)$$

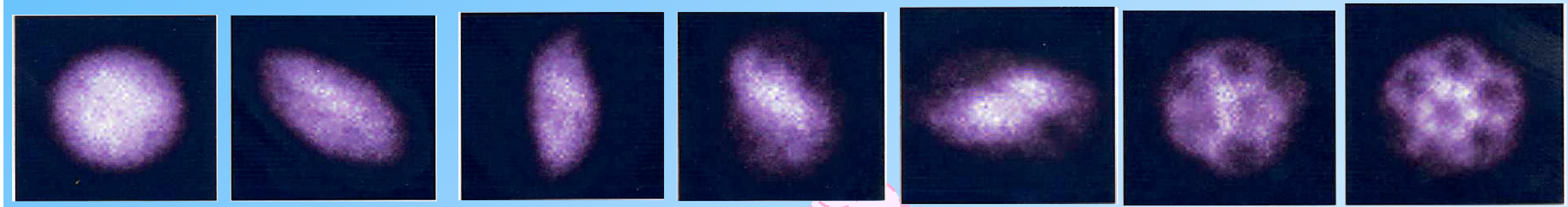
$$\varepsilon_x \neq \varepsilon_y$$



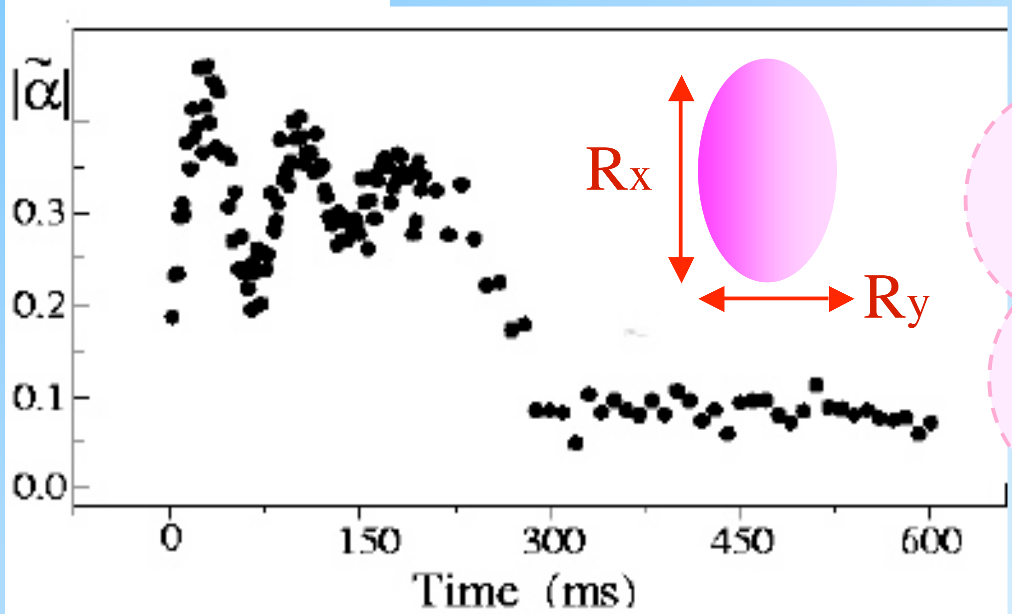
Direct observation of the vortex lattice formation

K.W. Madison *et al.* PRL **86** , 4443 (2001)

Snapshots of the BEC after turning on the rotation



$$\alpha = \Omega \frac{R_x^2 - R_y^2}{R_x^2 + R_y^2}$$



1. The BEC becomes elliptic, then oscillating.
2. The surface becomes unstable.
3. Vortices enter the BEC from the surface.
4. The BEC recovers the axisymmetry, the vortices forming a lattice.

The Gross-Pitaevskii(GP) equation in a rotating frame

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V_{\text{trap}}\Psi + g|\Psi|^2\Psi$$

Wave function $\Psi(\mathbf{r}, t)$

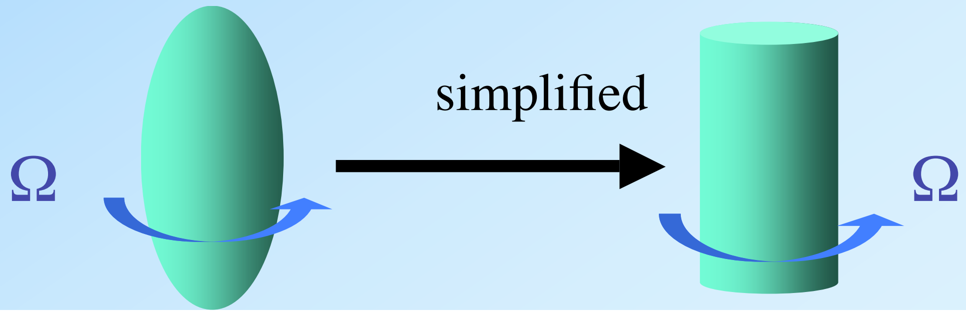
Interaction $g = \frac{4\pi\hbar^2 a_s}{m}$

a_s s-wave scattering length

in a rotating frame

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + (V_{\text{trap}} + U_{\text{stir}})\Psi + g|\Psi|^2\Psi - \Omega L_z \Psi$$

Two-dimensional $U_{\text{stir}}(\mathbf{R}) = \frac{m}{2}\omega_\perp^2(\varepsilon_x X^2 + \varepsilon_y Y^2)$



The GP equation with a dissipative term

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + (V_{\text{trap}} + U_{\text{stir}}) + g|\Psi|^2 - \mu - \Omega L_z \right] \Psi$$

$$(i - \gamma)\hbar \frac{\partial \Psi}{\partial t}$$

$\gamma = 0.03$: dimensionless parameter

S.Choi, et al. PRA 57, 4057 (1998)

I.Aranson, et al. PRB 54, 13072 (1996)

This dissipation comes microscopically from the interaction between the condensate and the noncondensate.

E.Zaremba, T. Nikuni, and A. Griffin, J. Low Temp. Phys. **116**, 277 (1999)

C.W. Gardiner, J.R. Anglin, and T.I.A. Fudge, J. Phys. B **35**, 1555 (2002)

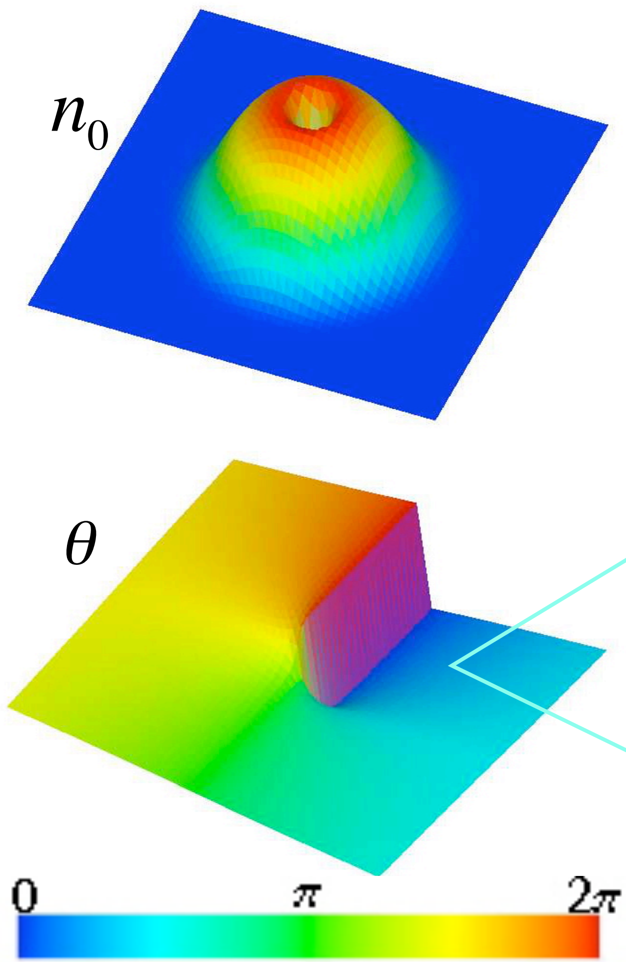
M. Kobayashi and M. Tsubota, Phys. Rev. Lett. **97**, 145301 (2006)

Profile of a single quantized vortex

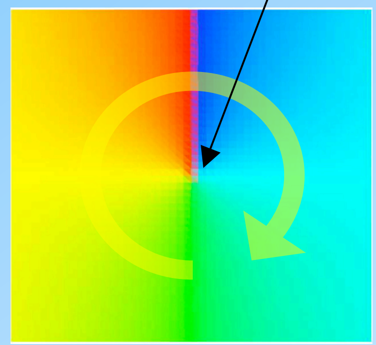
$$-\frac{\hbar^2}{2m}\nabla^2\Psi + V_{\text{trap}}\Psi + g|\Psi|^2\Psi = \mu\Psi$$

$$\Psi(r) = \sqrt{n_0(r)}e^{i\theta(r)}$$

A quantized vortex

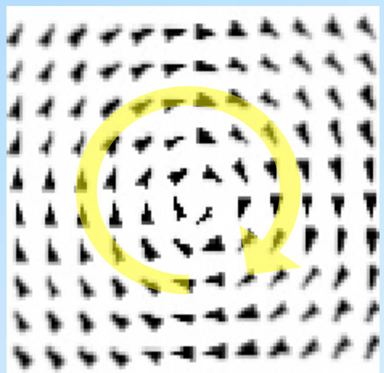


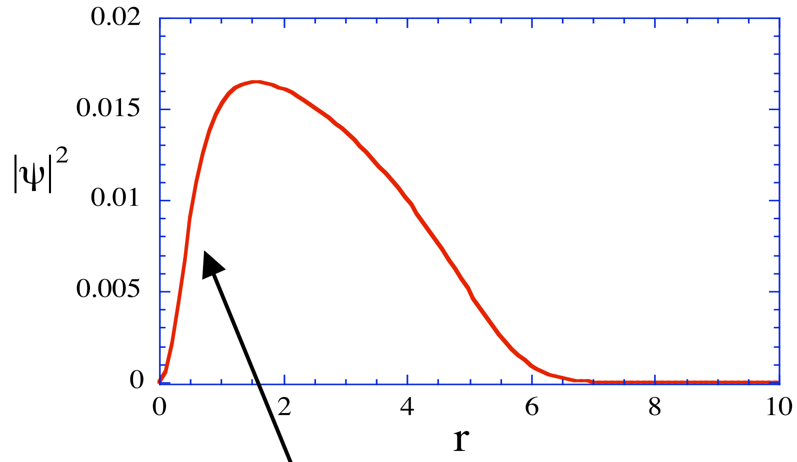
A vortex



$$\mathbf{v}_s \equiv \frac{\hbar}{m}\nabla\theta$$

Velocity field





Vortex core= healing length

$$\xi \approx \frac{\hbar}{\sqrt{2mgn_0}}$$

Dynamics of the vortex lattice formation (1)

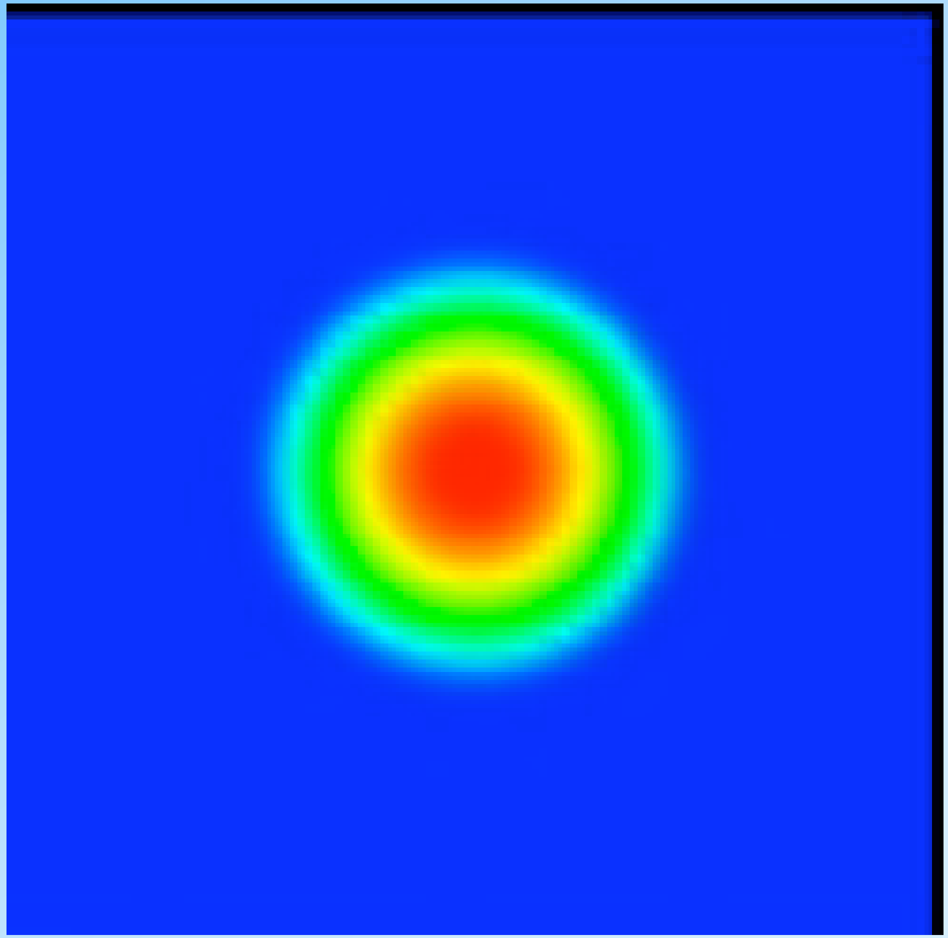
Time development of the condensate density n_0

$$\Omega = 0.7\omega_\perp$$

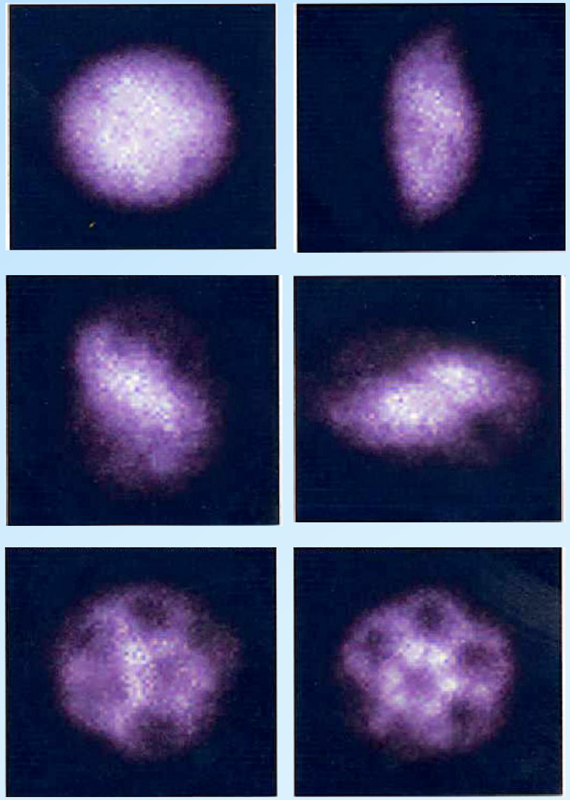
MT, K. Kasamatsu, M. Ueda, Phys. Rev. A **65**, 023603 (2002)

$$V_{\mathrm{trap}}(r) = \frac{1}{2} m\omega_\perp^2 r^2$$

$$\Psi(r) = \sqrt{n_0(r)} e^{i\theta(r)}$$



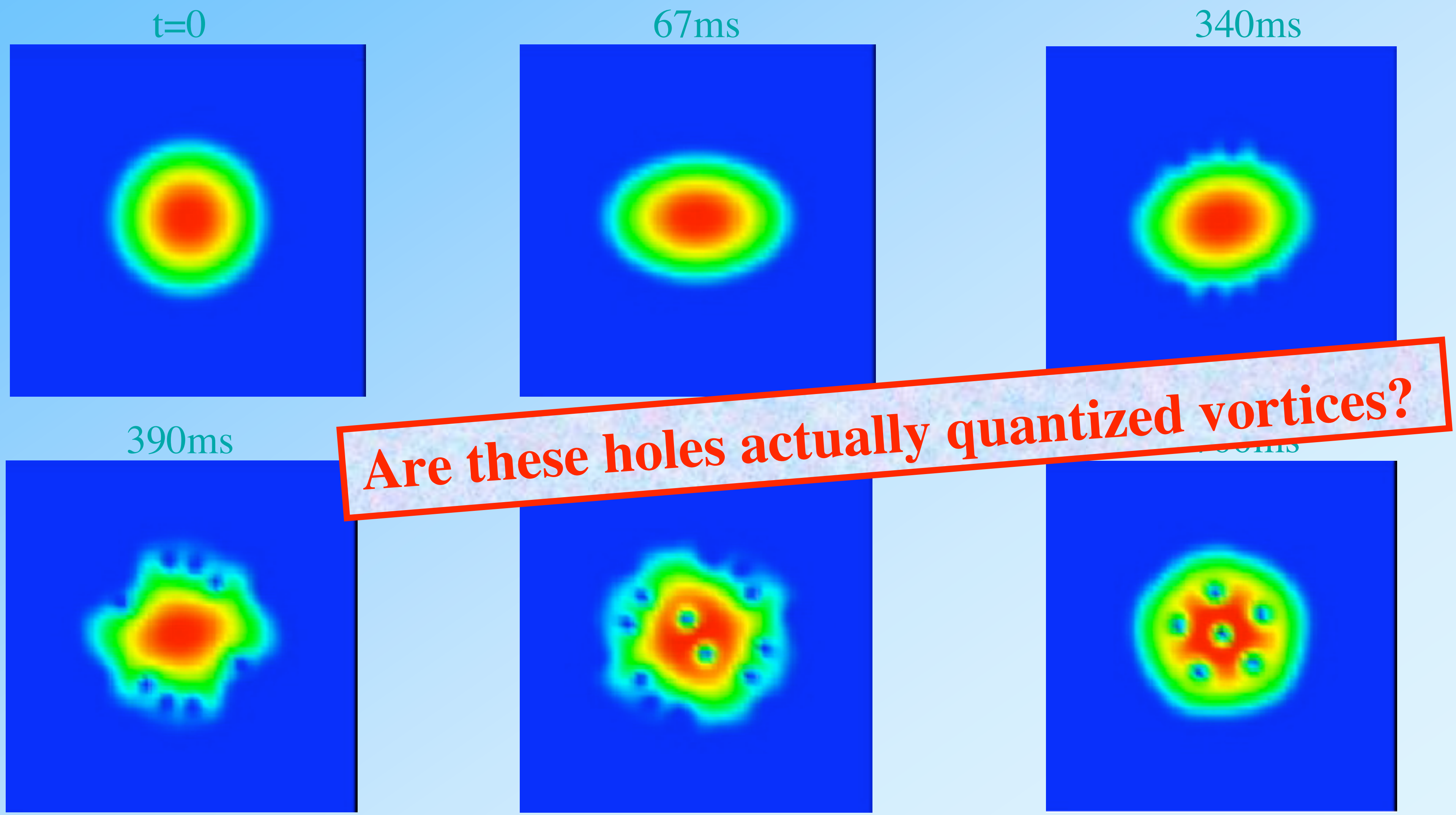
Experiment



K.W.Madison *et al.* PRL **86** , 4443 (2001)

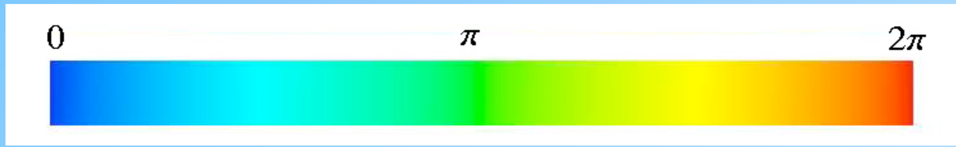
Dynamics of the vortex lattice formation (2)

Time-development of the condensate density n_0

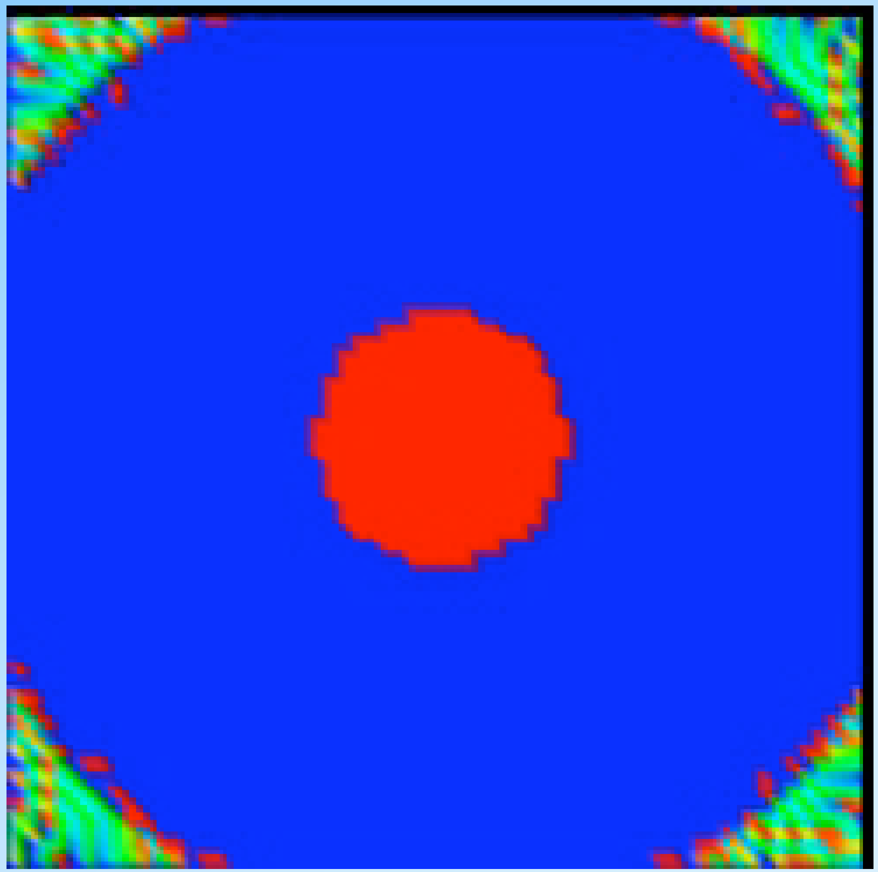


Dynamics of the vortex lattice formation (3)

Time-development of the phase θ

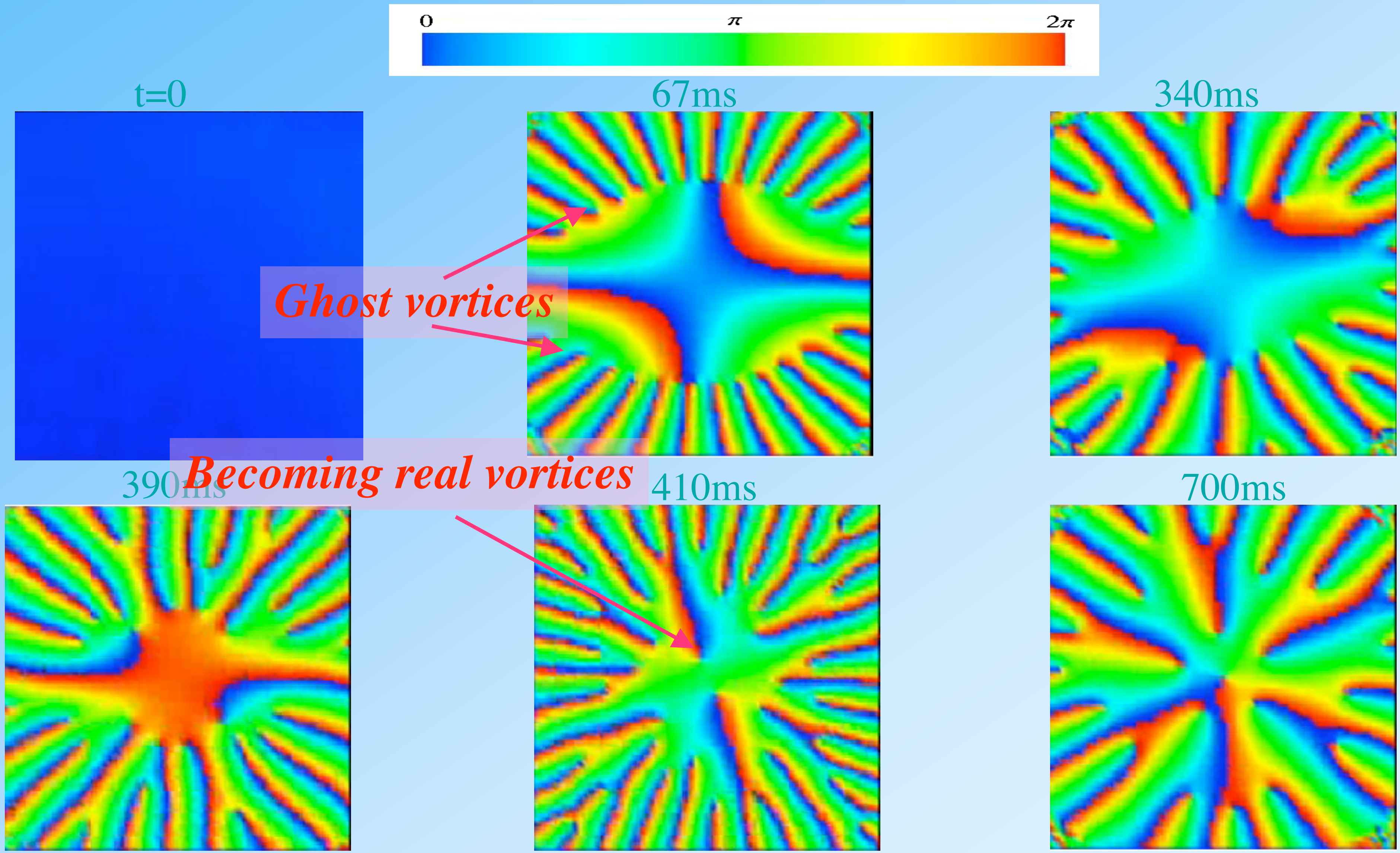


$$\Psi(r) = \sqrt{n_0(r)}e^{i\theta(r)}$$

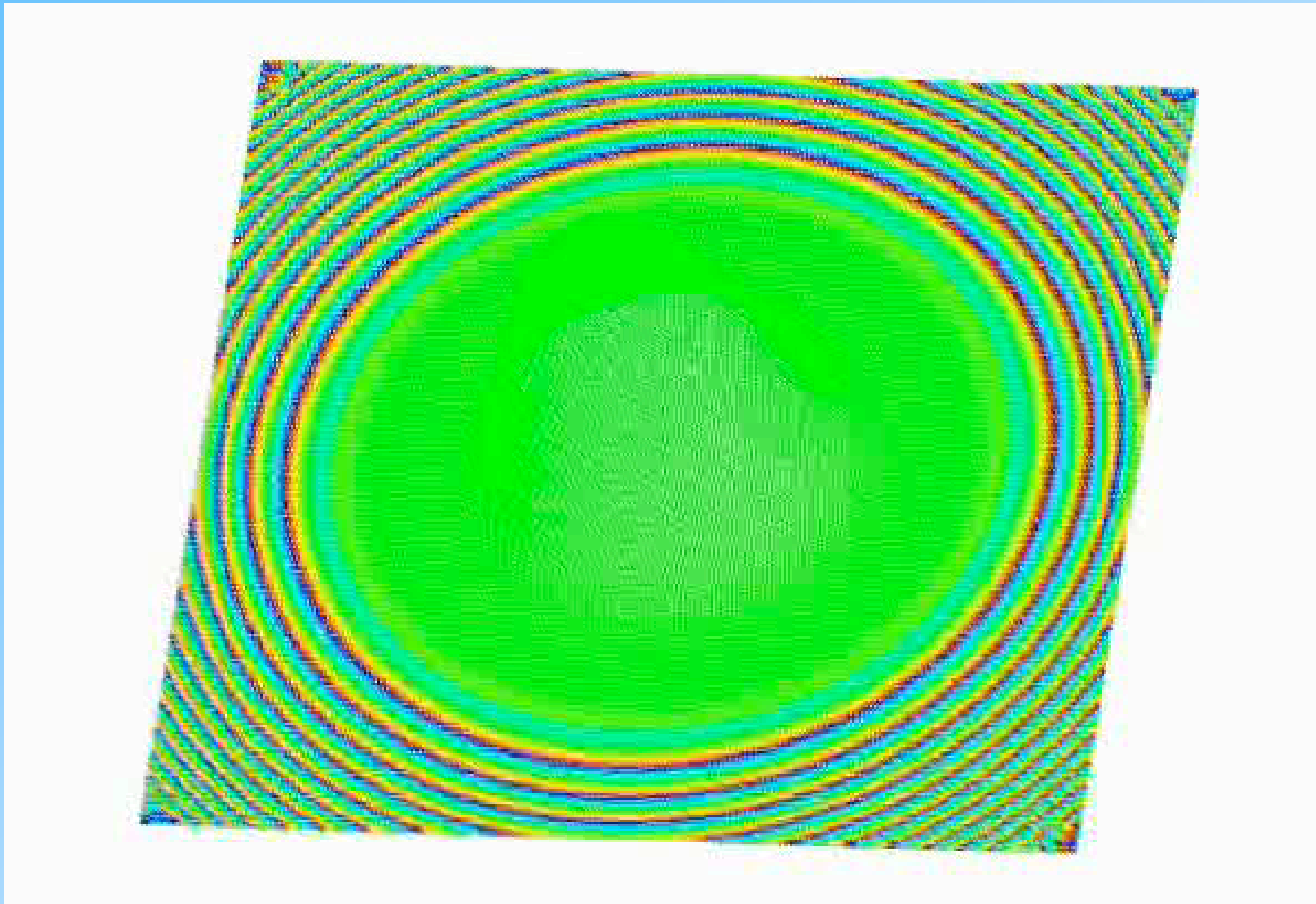


Dynamics of the vortex lattice formation (4)

Time-development of the phase θ



Simultaneous display of the density and the phase



M.Tsubota, K.Kasamatsu, M.Ueda, Phys.Rev.A65, 023603(2002)
K.Kasamatsu, T.Tsubota, M.Ueda, Phys.Rev.A67, 033610(2003)

Contents

2. QT by the GP model

M. Kobayashi, MT, Phys. Rev. Lett. 94, 065302 (2005):
J. Phys. Soc. Jpn.74, 3248 (2005)

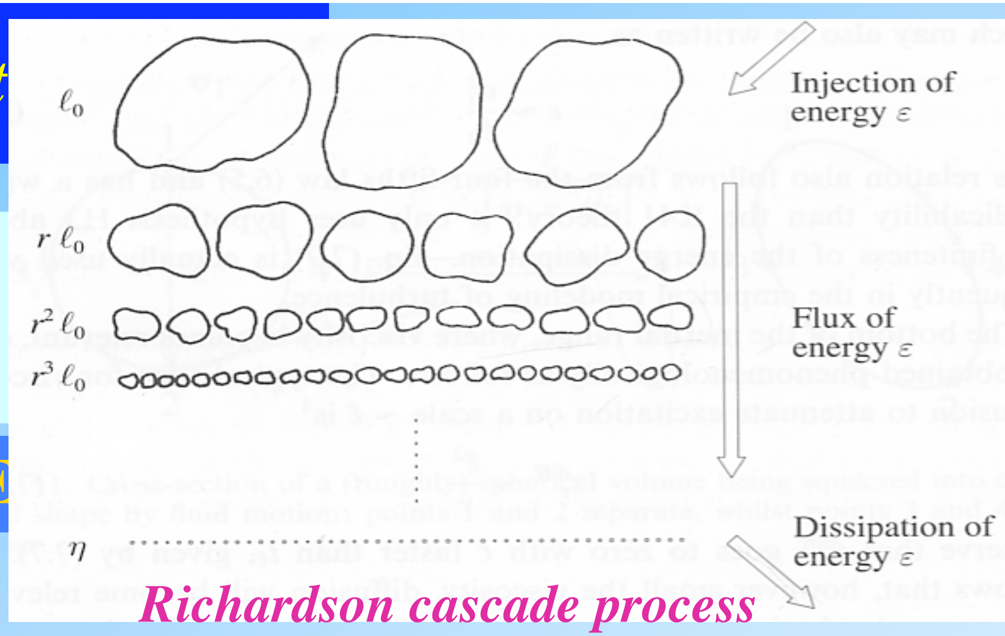
The main interests

What is the relation between QT and CT?

How similar? How different?

We will focus on the statistical quantities.

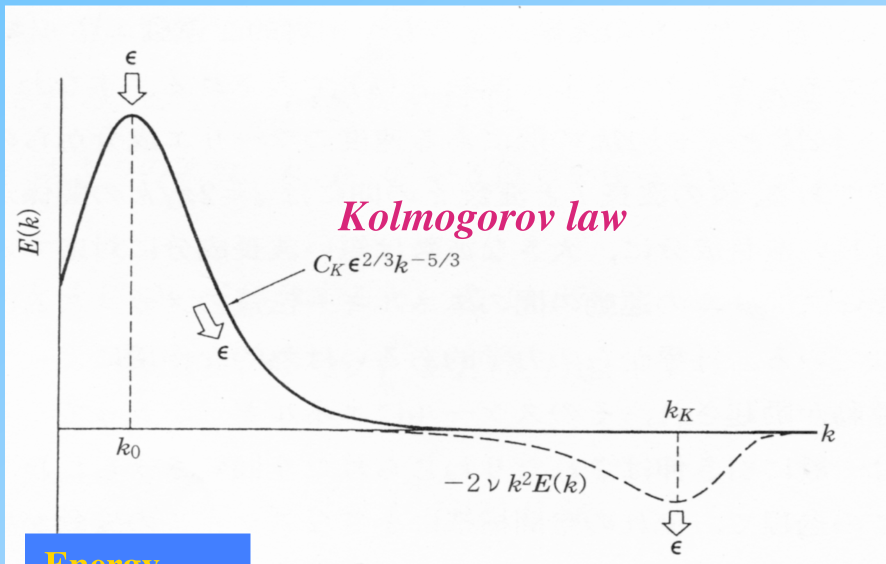
Energy spectra of fully developed t



Richardson cascade process

Energy spectrum of the velocity field

$$E = \frac{1}{2}\int \mathbf{v}^2 \, d\mathbf{r} = \int E(k)\,dk$$



Energy-containing range | Inertial range | Energy-dissipative range

Energy spectrum of turbulence

The .

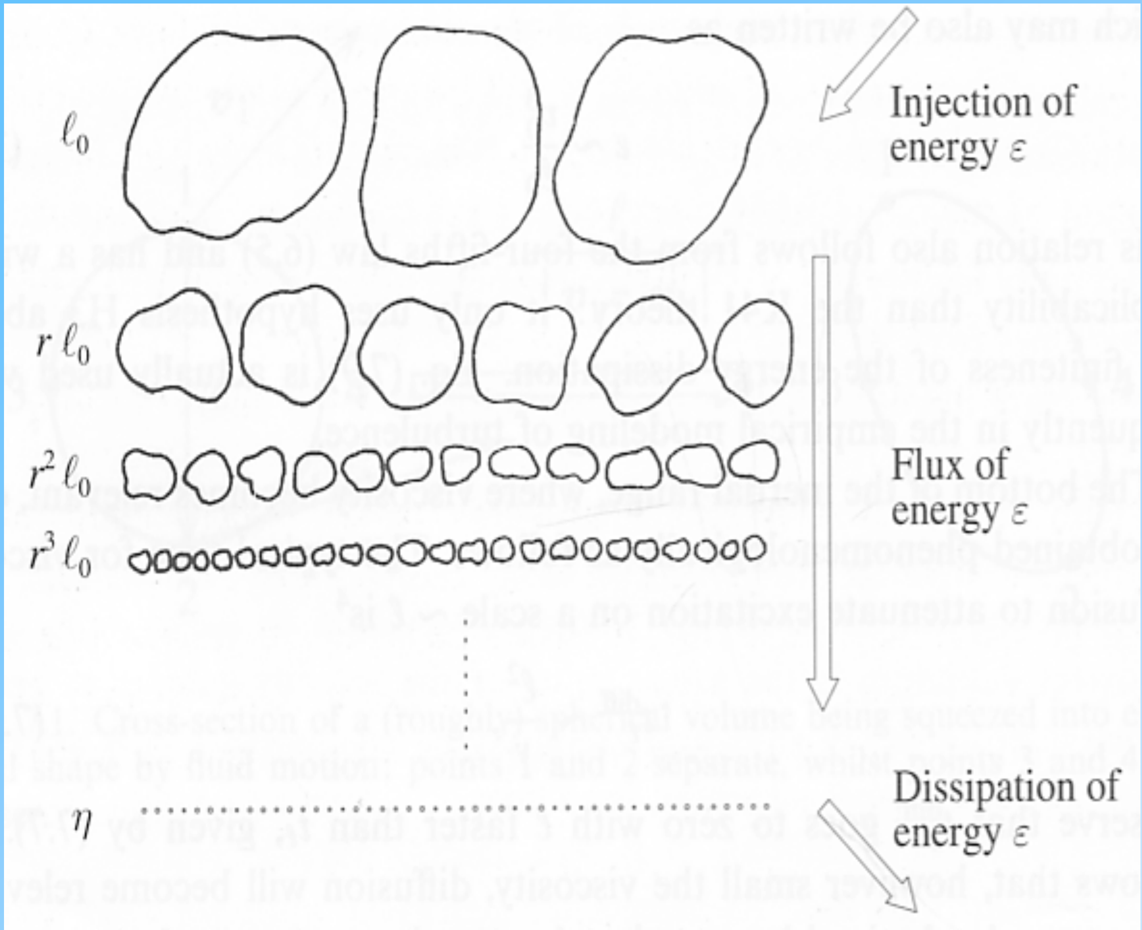
Inertial range

Dissipation does not work. The nonlinear interaction transfers the energy from low *k* region to high *k* region.

Kolmogorov law (K41) : $E(k)=C\varepsilon^{2/3} k^{-5/3}$

Energy-dissipative range

The energy is dissipated with the rate ε at the Kolmogorov wave number $k_c = (\varepsilon/\nu^3)^{1/4}$.



Most textbooks describe this kind of Richardson cascade. However, this is only a cartoon; nobody has ever confirmed it clearly.

One of the reasons is that it is not so easy to identify each eddy in a fluid.

Does quantum turbulence (QT) satisfy the Kolmogorov law?

If so, QT shows some analogy with CT.

Having this sort of motivation, the studies of QT have entered a new stage since the middle of 90's !

Energy spectra of quantum turbulence (QT)

There are three works which directly study the energy spectrum of QT at zero temperature.

Decaying Kolmogorov turbulence in a model of superflow

C. Nore, M. Abid and M.E.Brachet, Phys.Fluids 9, 2644 (1997)

The Gross-Pitaevskii (GP) model

Energy Spectrum of Superfluid Turbulence with No Normal-Fluid Component

T. Araki, M.Tsubota and S.K.Nemirovskii, Phys.Rev.Lett.89, 145301(2002)

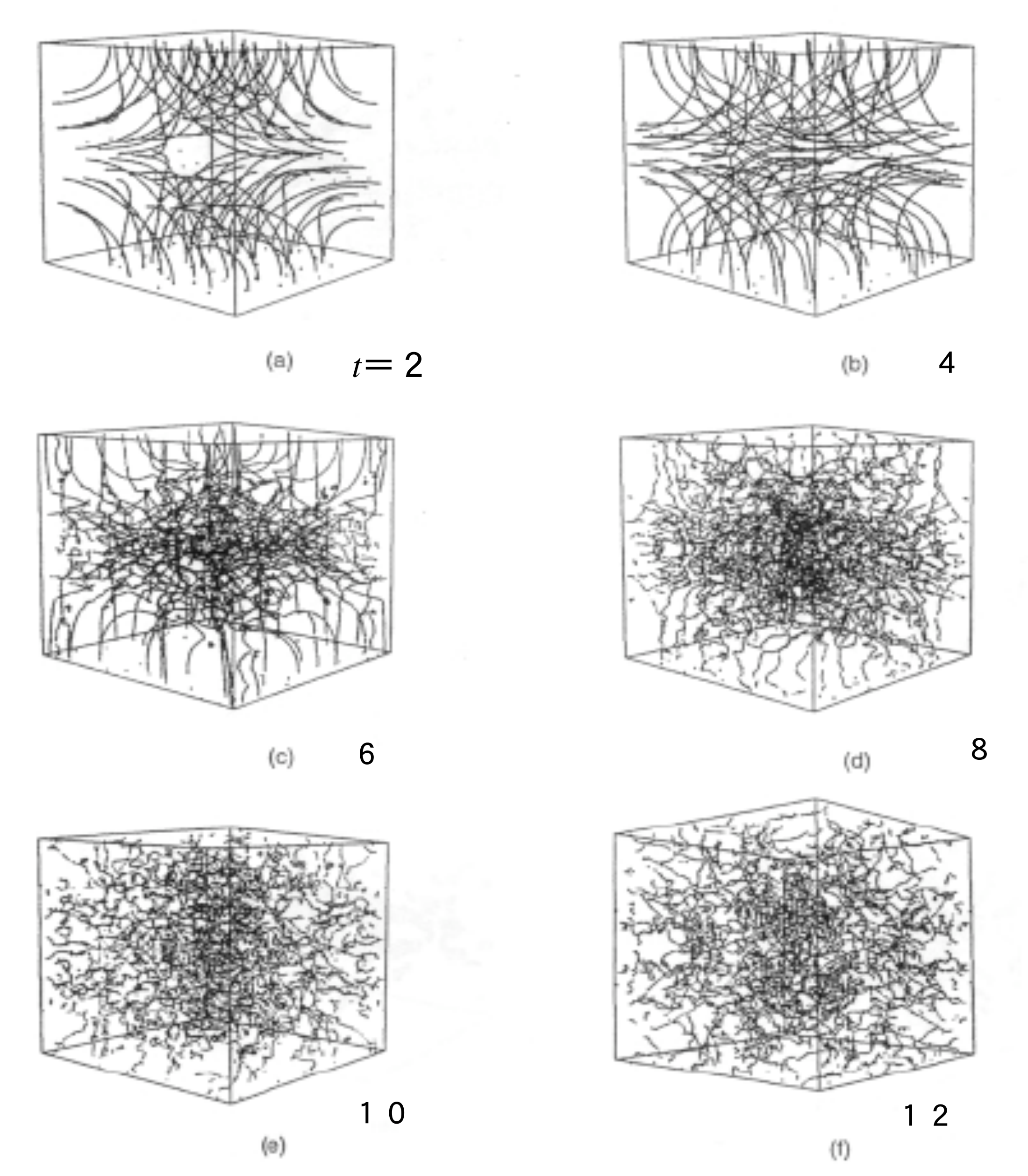
The vortex-filament model

Kolmogorov Spectrum of Superfluid Turbulence: Numerical Analysis of the Gross-Pitaevskii Equation with a Small-Scale Dissipation

M. Kobayashi and M. Tsubota,

Phys. Rev. Lett. 94, 065302 (2005), J. Phys. Soc. Jpn.74, 3248 (2005).

C. Nore, M. Abid and M.E.Brachet, Phys.Fluids 9, 2644(1997)



By using the GP model, they obtained a vortex tangle with starting from the Taylor-Green vortices.

We should note that the GP model describes a compressible fluid.

In order to study the Kolmogorov spectrum, it is necessary to decompose the total energy into some components. (Nore *et al.*, 1997)

Total energy

$$E = \frac{1}{\int d\mathbf{x}\,\rho}\int d\mathbf{x}\,\Phi^*\left[-\nabla^2 + \frac{g}{2}|\Phi|^2\right]\Phi \qquad \Phi = \sqrt{\rho}\exp(i\theta)$$

$$E = E_{\mathrm{int}} + E_q + E_{kin}$$

The kinetic energy $E_{kin} = \frac{1}{\int d\mathbf{x}\,\rho}\int d\mathbf{x}\left(|\Phi|\nabla\theta\right)^2$ is divided into

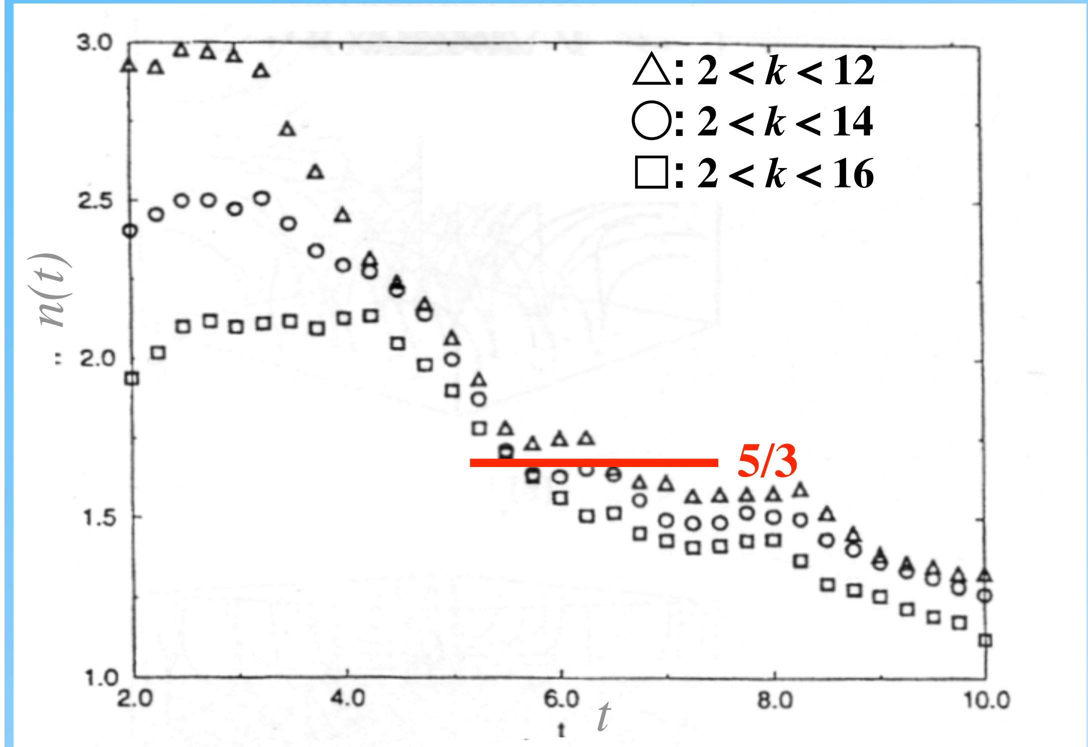
the compressible part $E_{kin}{}^c = \frac{1}{\int d\mathbf{x}\,\rho}\int d\mathbf{x}\left[\left(|\Phi|\nabla\theta\right)^c\right]^2$ with $\mathrm{rot}\left(|\Phi|\nabla\theta\right)^c = 0$

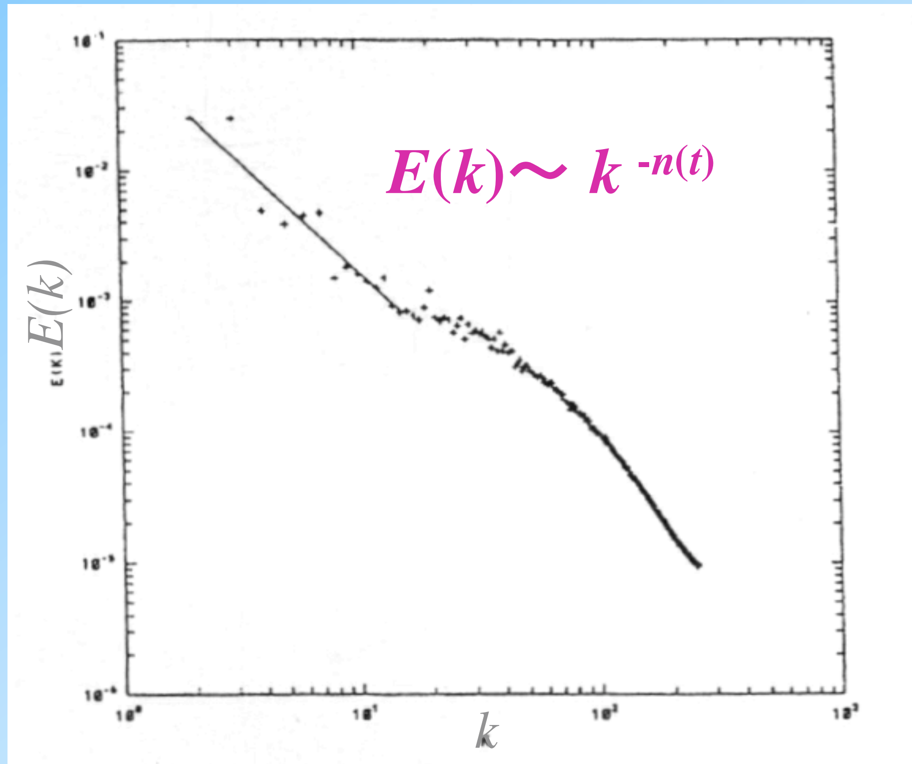
and

the incompressible part $E_{kin}{}^i = \frac{1}{\int d\mathbf{x}\,\rho}\int d\mathbf{x}\left[\left(|\Phi|\nabla\theta\right)^i\right]^2$ with $\mathrm{div}\left(|\Phi|\nabla\theta\right)^i = 0$.

This incompressible kinetic energy $E_{\mathrm{kin}}{}^{\mathrm{i}}$ should obey the Kolmogorov spectrum.

C. Nore, M. Abid and M.E.Brachet, Phys.Fluids 9, 2644(1997)

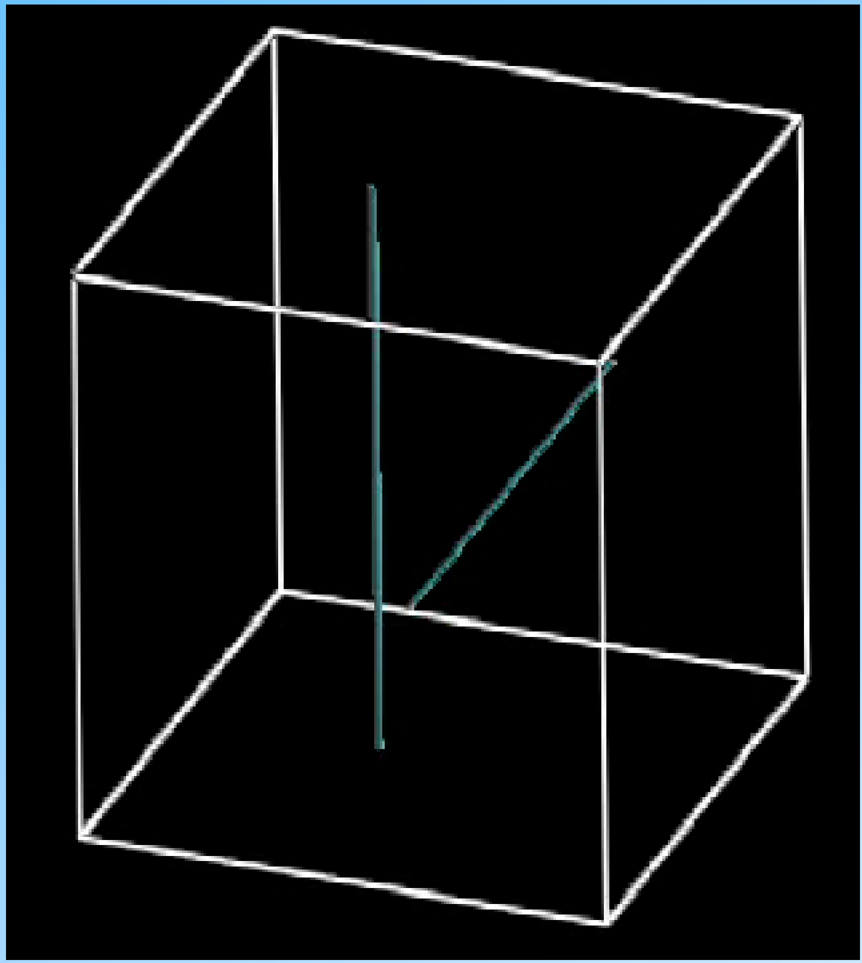


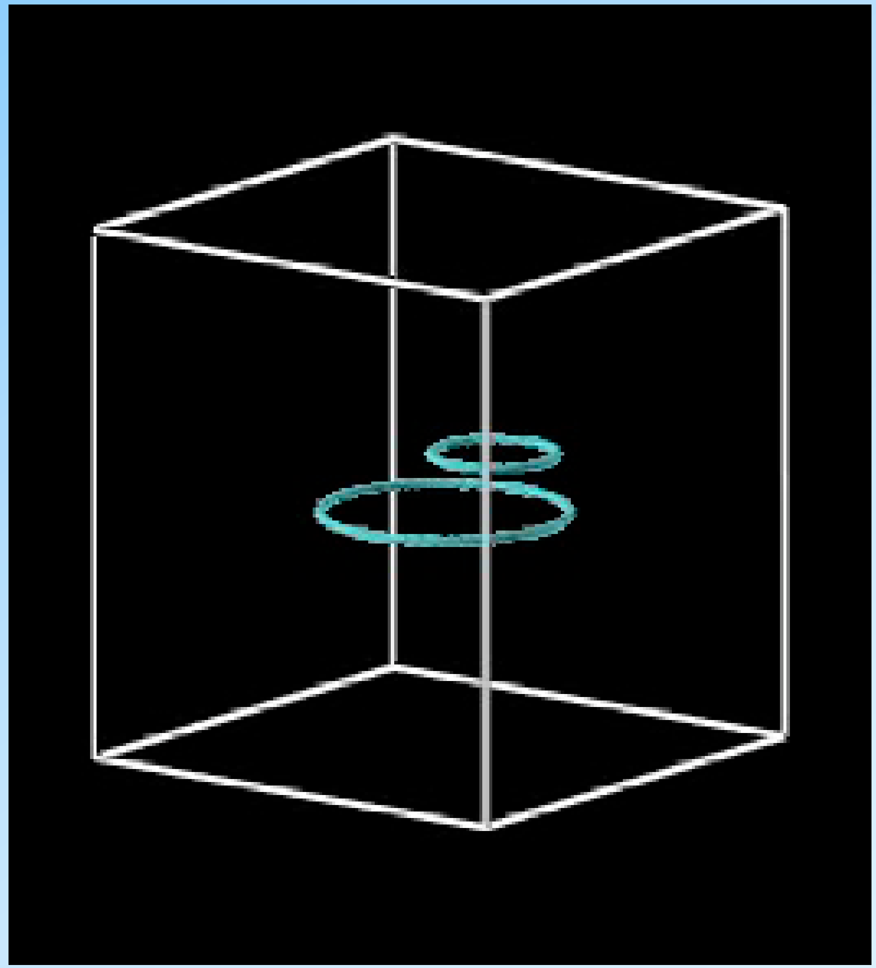


The right figure shows the energy spectrum at a moment. The left figure shows the development of the exponent *n(t)*. The exponent *n(t)* goes through 5/3 on the way of the dynamics.

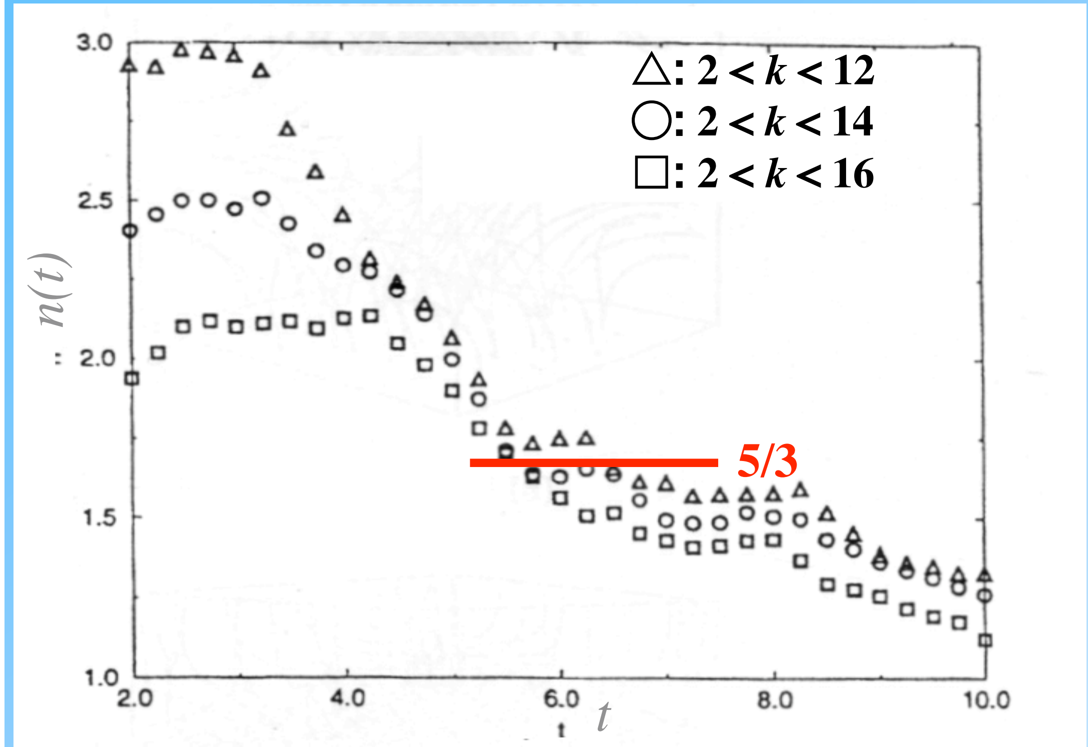
In the late stage, however, the exponent deviates from 5/3, because the sound waves resulting from vortex reconnections disturb the cascade process of the inertial range.

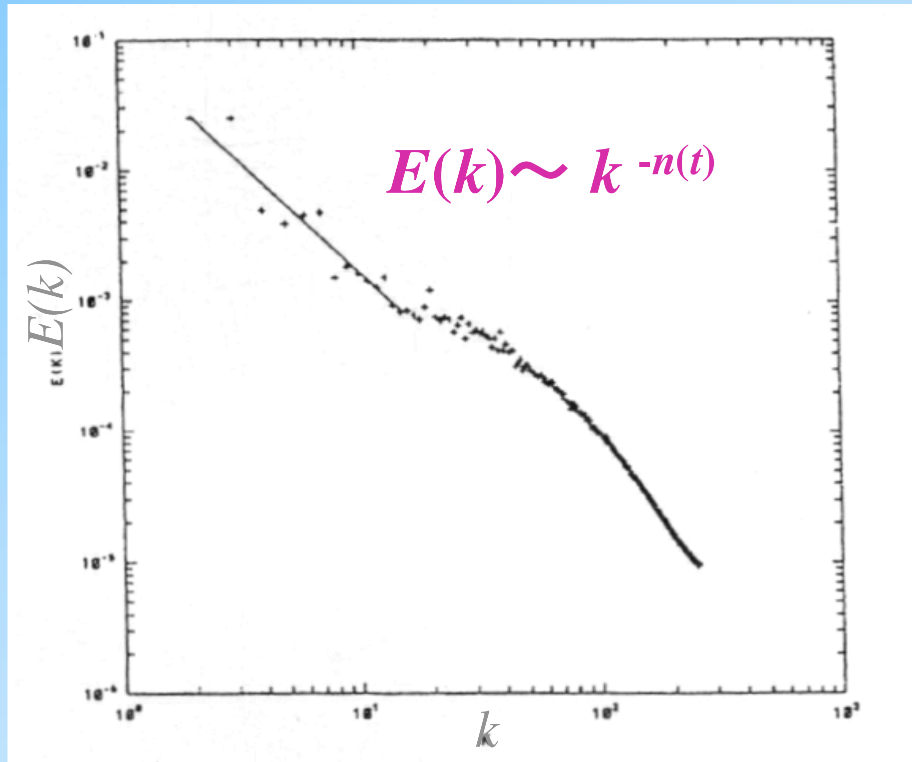
Reconnection of quantized vortices
(Numerical analysis of the Gross-Pitaevskii equation)





C. Nore, M. Abid and M.E.Brachet, Phys.Fluids 9, 2644(1997)





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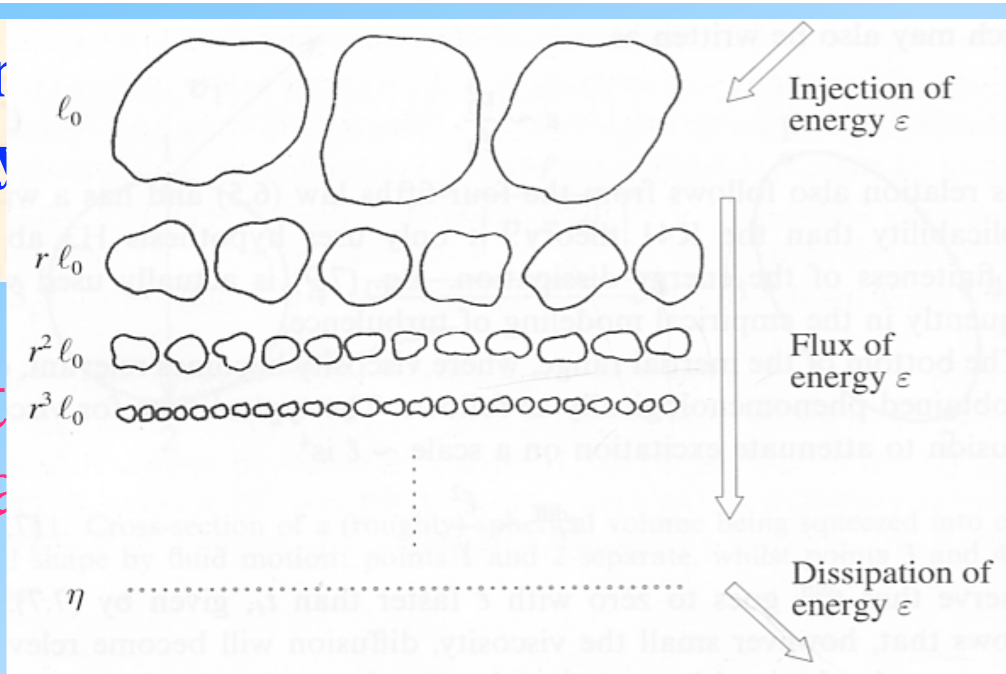
Kolmogorov spectrum of quan

M. Kobayashi and M. Tsubota, Phy

J. Phys. Soc. Jpn. 74, 3248 (2005)



1. We solved the GP equat space in order to use the fa transformation.

2. We made a steady state of turbulence. In order to do that,

2-1 We introduced a dissipative term which dissipates the Fourier component of the high wave number, namely, phonons of short wave length.

2-2 We excited the system at a large scale by moving a random potential.

The GP equation in the real space

$$i\frac{\partial}{\partial t}\Phi(\mathbf{r},t) = \left[-\nabla^2 - \mu + g\left|\Phi(\mathbf{r},t)\right|^2\right]\Phi(\mathbf{r},t)$$

The GP equation in the Fourier space

$$i\frac{\partial}{\partial t}\Phi(\mathbf{k},t) = \left(k^2 - \mu\right)\Phi(\mathbf{k},t)$$

$$+\frac{g}{V^2}\sum_{\mathbf{k}_1,\mathbf{k}_2}\Phi(\mathbf{k}_1,t)\,\Phi^*(\mathbf{k}_2,t)\,\Phi(\mathbf{k}-\mathbf{k}_1+\mathbf{k}_2,t)$$

$\xi^2 = 1/g|\Phi|^2$ healing length giving the vortex core size

To solve the GP equation numerically with high accuracy, we use the Fourier spectral method in space with the periodic boundary condition in a cube.

How to dissipate the energy at small scales?

The GP equation with the small scale dissipation

$$\{i-\gamma(k)\}\frac{\partial}{\partial t}\Phi(\mathbf{k},t)=\left(k^2-\mu\right)\Phi(\mathbf{k},t)$$

$$+\frac{g}{V^2}\sum_{\mathbf{k}_1,\mathbf{k}_2}\Phi(\mathbf{k}_1,t)\Phi^*(\mathbf{k}_2,t)\Phi(\mathbf{k}-\mathbf{k}_1+\mathbf{k}_2,t)$$

$\xi^2=1/g|\Phi|^2$:healing length giving the vortex core size

$$\gamma(k)=\gamma_0\theta(k-2\pi/\xi)$$

We introduce the dissipation that works only in the scale smaller than ξ.

Since there is no vortex motion at the scales smaller than ξ, this dissipation must work for only short-wavelength sound waves.

cf. M. Kobayashi and M. Tsubota, PRL **97**, 145301 (2006)

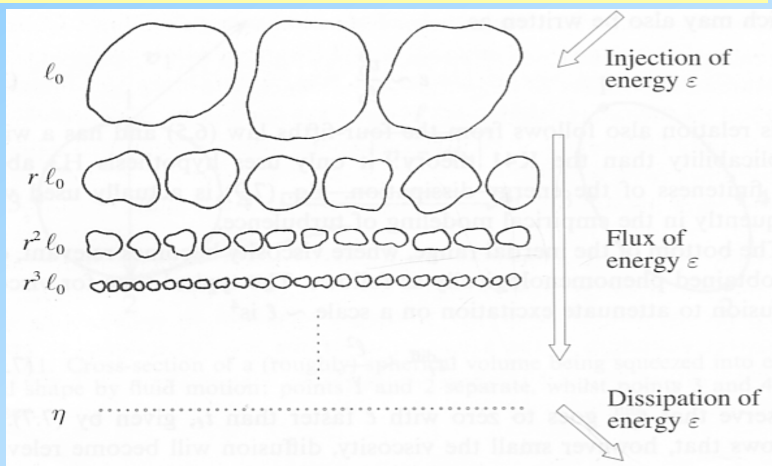
How to inject the energy at large scales?

This is done by moving the random potential satisfying the space-time correlation:

$$\left\langle V(\mathbf{x},t)V(\mathbf{x}',t')\right\rangle = V_0^2 \exp\left[-\frac{(\mathbf{x}-\mathbf{x}')^2}{2X_0^2} - \frac{(t-t')^2}{2T_0^2}\right]$$

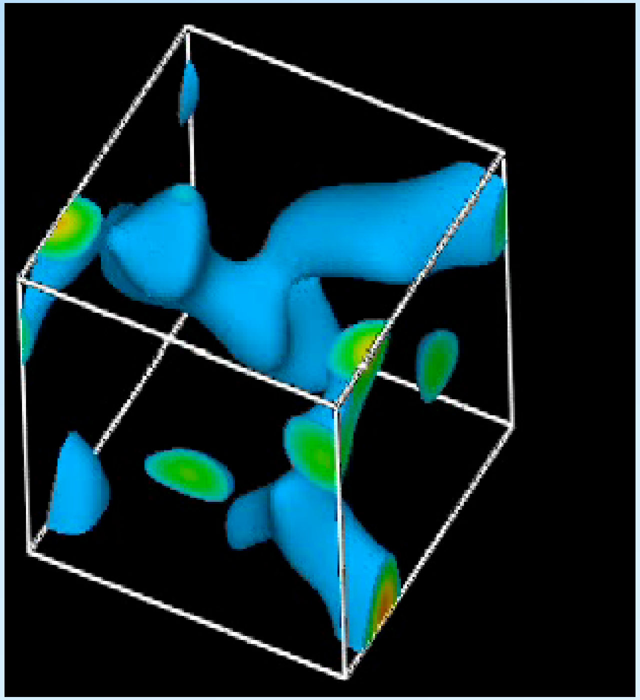
The variable X_0 determines the scale of the energy-containing range.

V_0=50, X_0=4 and T_0=6.4×10^{-2}



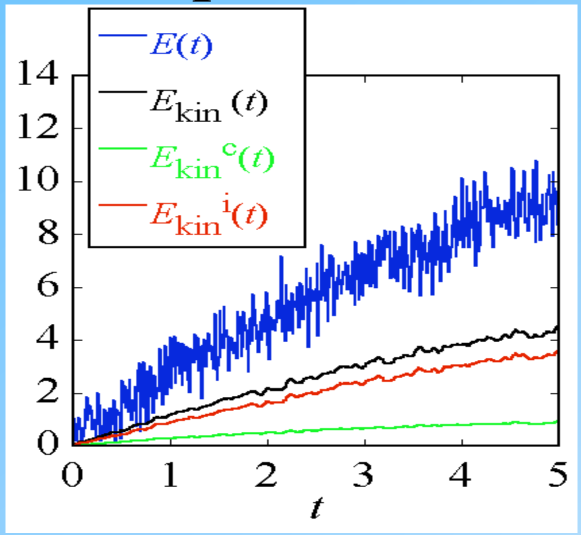
X_0

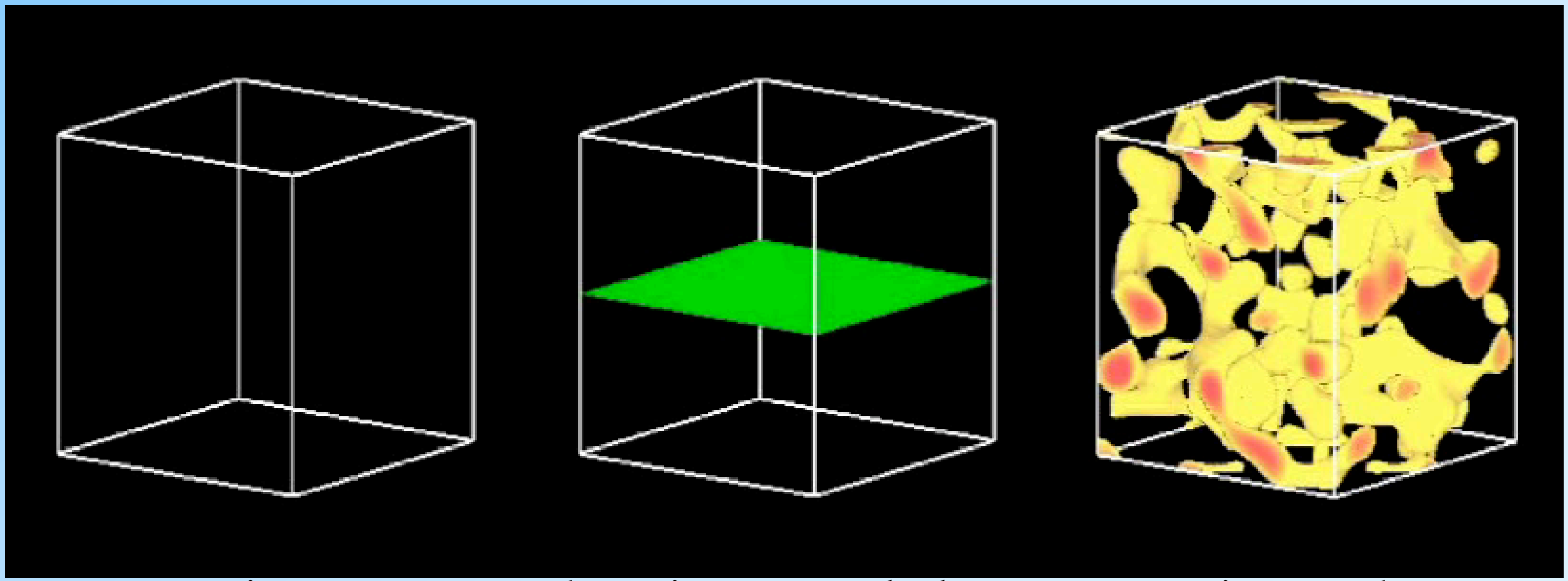
ξ



Thus steady turbulence is obtained.(1)

Time development of each energy component

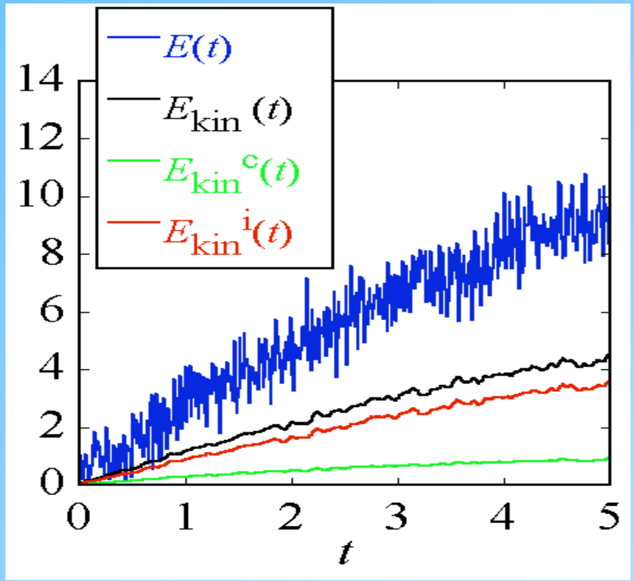


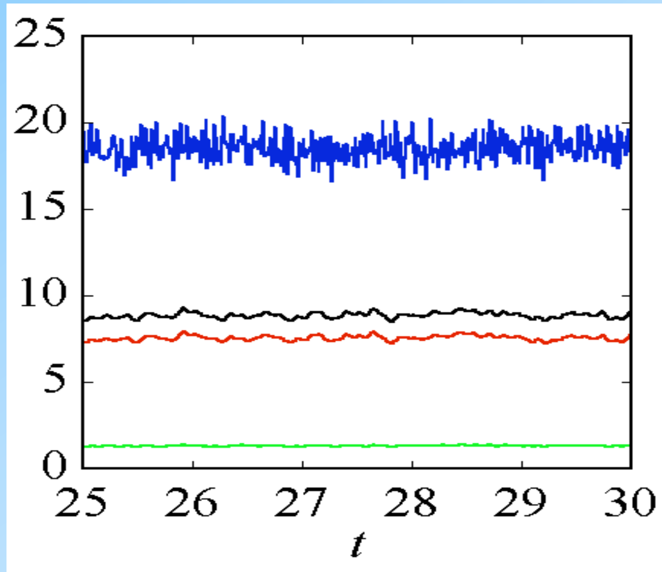


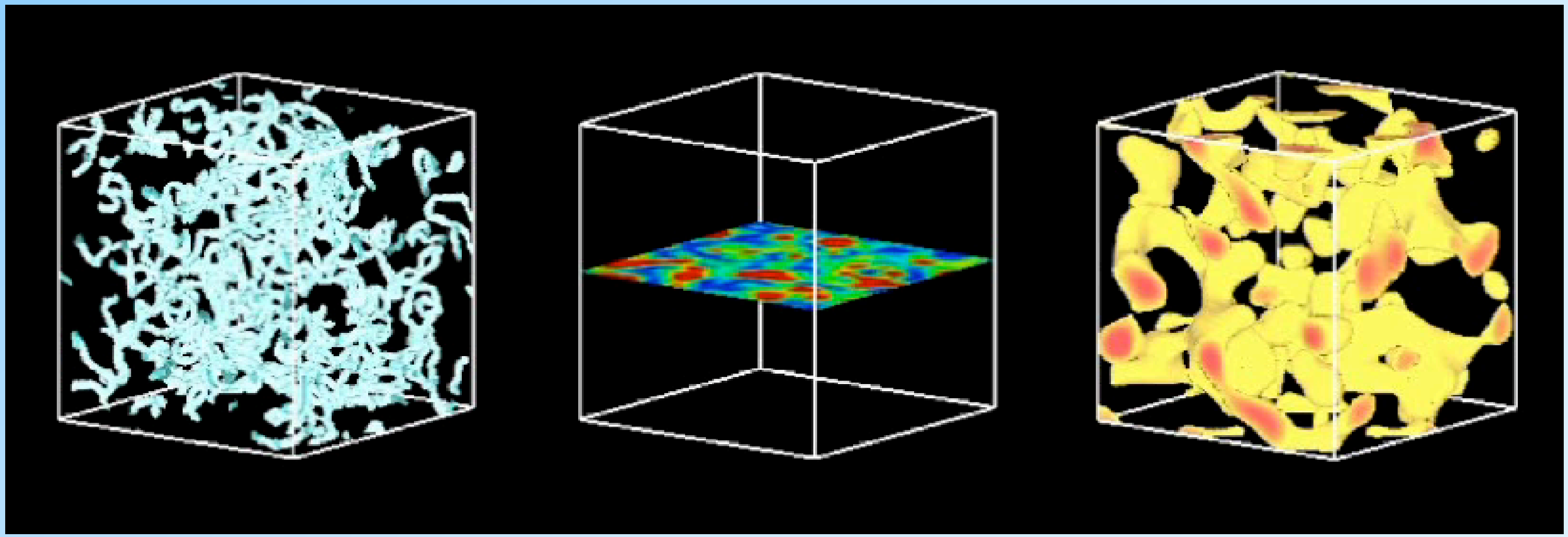
Vortices Phase in a central plane Moving random potential

Thus steady turbulence is obtained.(2)

Time development of each energy component

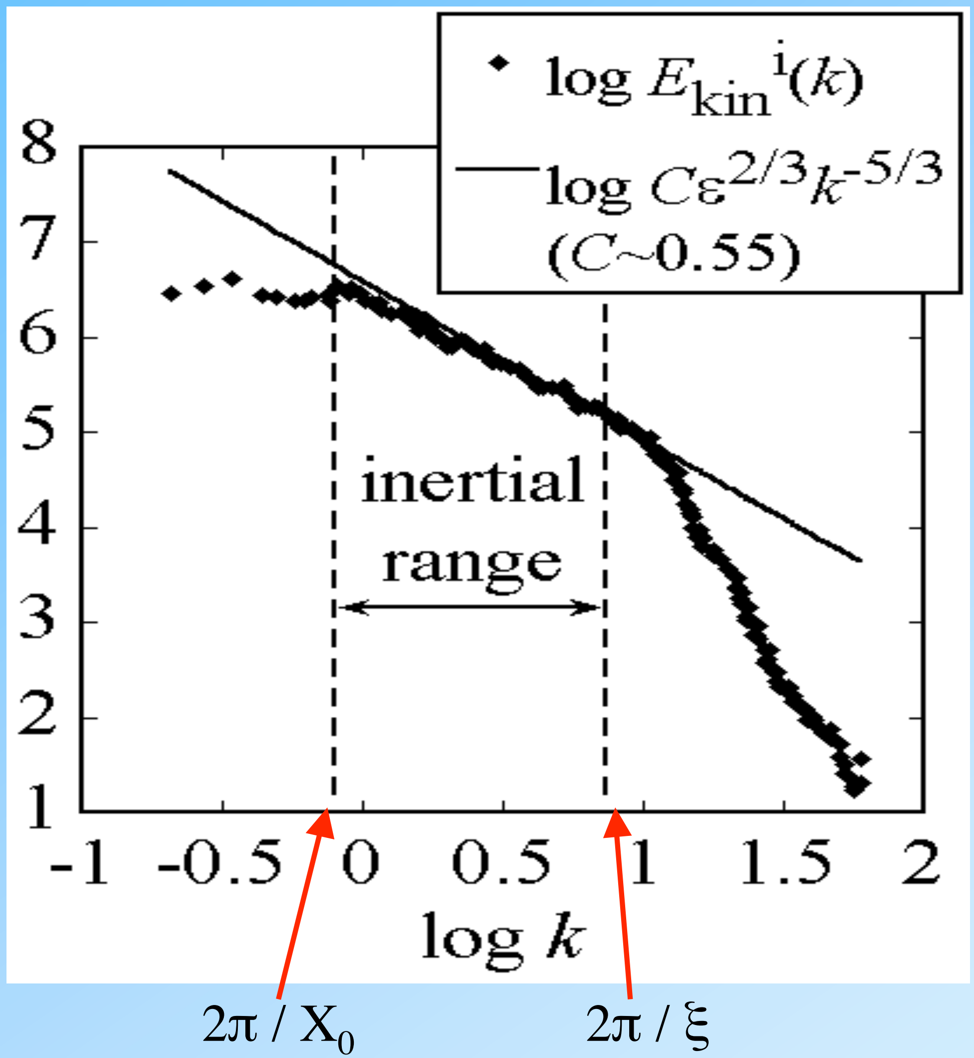






Energy spectrum of the steady turbulence

The energy spectrum obeys the Kolmogorov form.

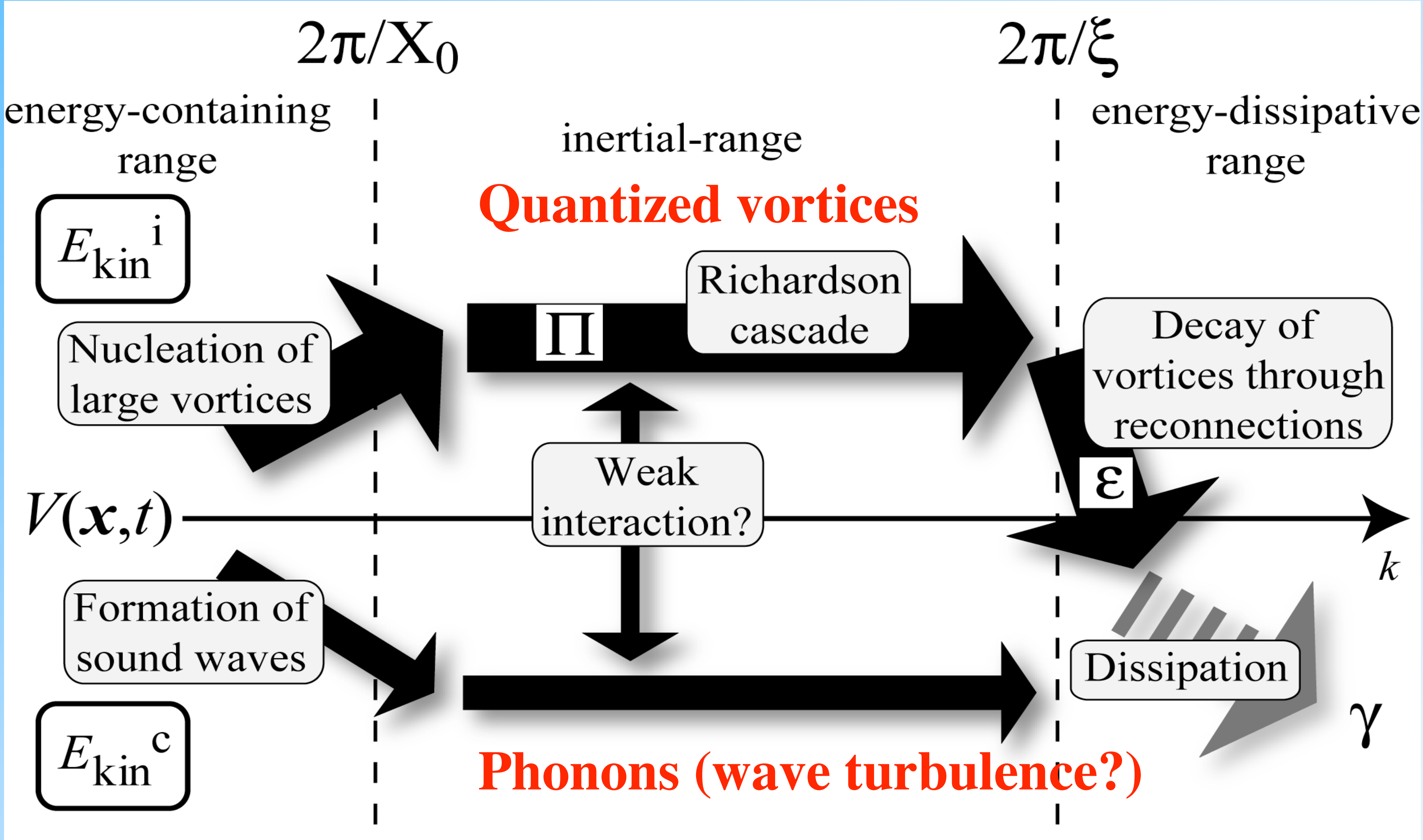


Quantum turbulence is found to show the essence of classical turbulence!

The inertial range is sustained by the Richardson cascade of quantized vortices.

M. Kobayashi and M. Tsubota, J. Phys. Soc. Jpn. 74, 3248 (2005)

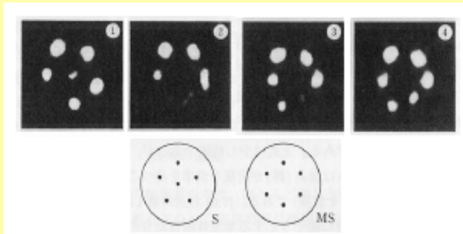
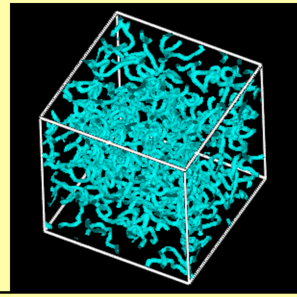
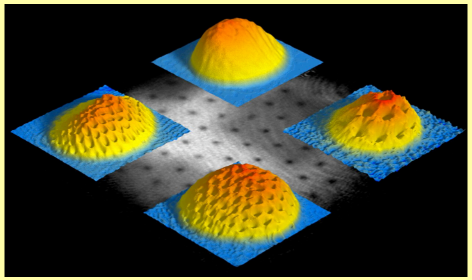
Picture of the cascade process



Contents

3. QT in Atomic Bose-Einstien condensates (BECs)

There are two main cooperative phenomena of quantized vortices; Vortex lattice under rotation and Vortex tangle (Quantum turbulence).

	Vortex lattice	Vortex tangle
Superfluid He		
Atomic BEC		None

Is it possible to make turbulence in a trapped BEC?

(1) We cannot apply some dc flow to the system.

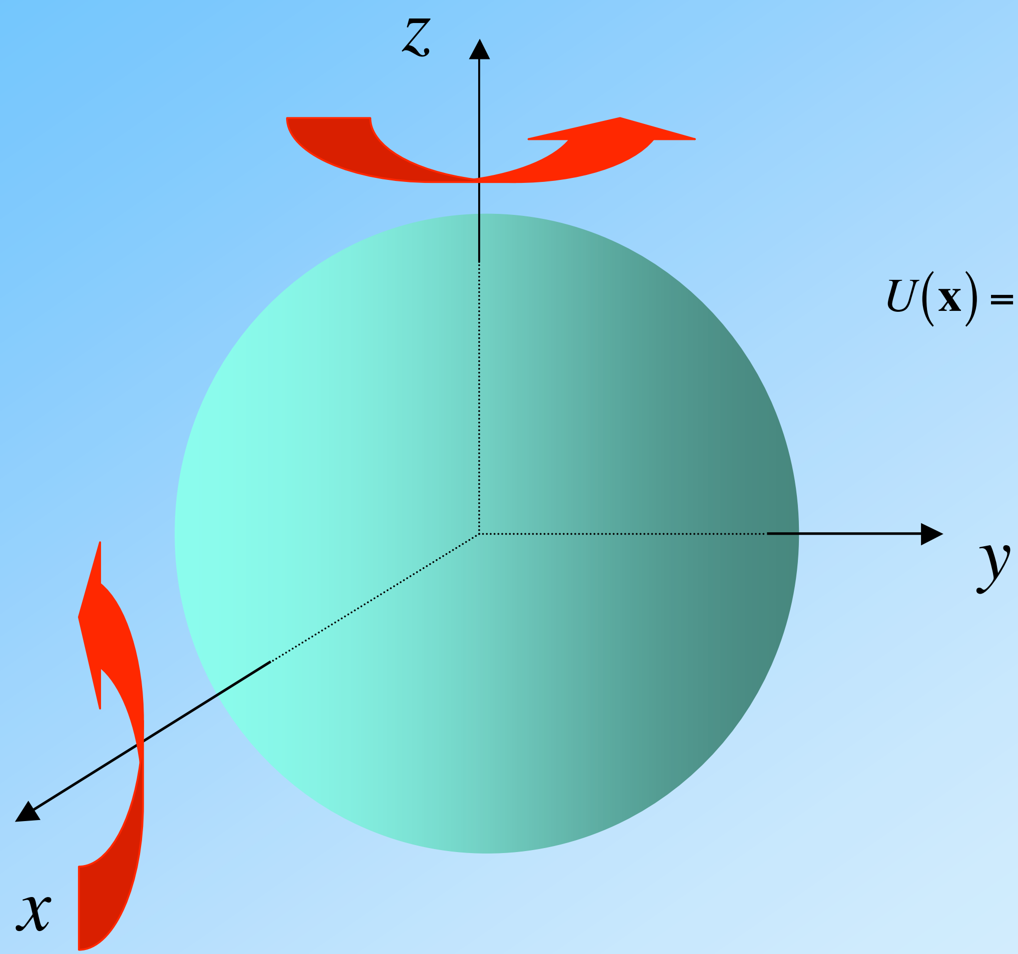
(2) This is a finite-size trapped system. Is this serious?

We use the idea of rotating turbulence.

QT in a trapped BEC

M. Kobayashi and M. Tsubota, Phys. Rev. A76, 045603 (2007)

Making QT by combining two rotations



1. Trap the BEC in a weakly elliptic potential.

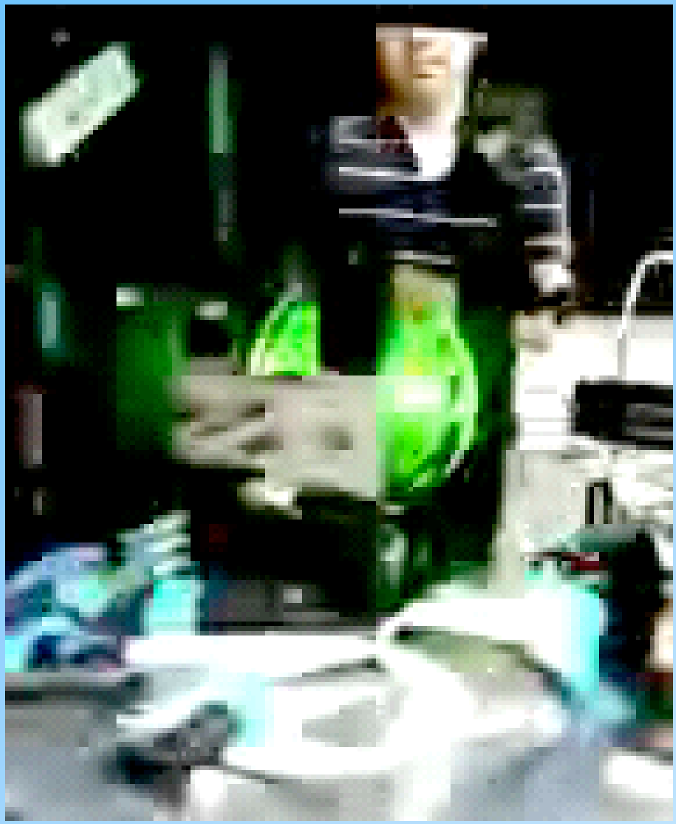
$$U(\mathbf{x}) = \frac{m\omega^2}{2}\left[(1-\varepsilon_1)(1-\varepsilon_2)x^2 + (1+\varepsilon_1)(1-\varepsilon_2)y^2 + (1+\varepsilon_2)z^2\right]$$

2. Rotate the system first around the x-axis, next around the z-axis.

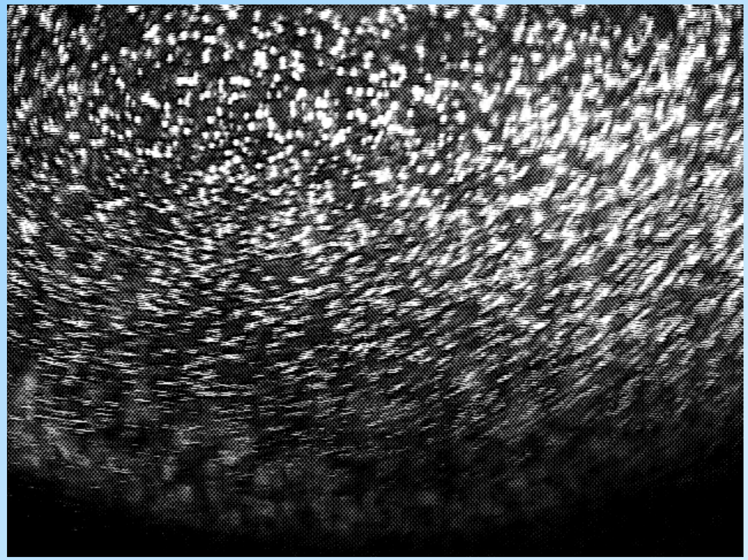
$$\Omega(t) = (\Omega_x,\ \Omega_z \sin\Omega_x t, \Omega_z \cos\Omega_x t)$$

Actually this idea has been already used in CT.

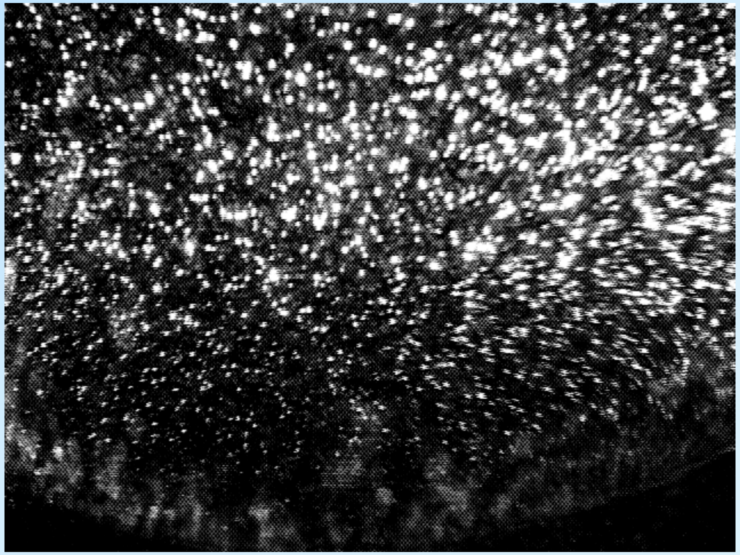
S. Goto, N. Ishii, S. Kida, and M. Nishioka, Phys. Fluids 19, 061705 (2007)



Rotation around one axis



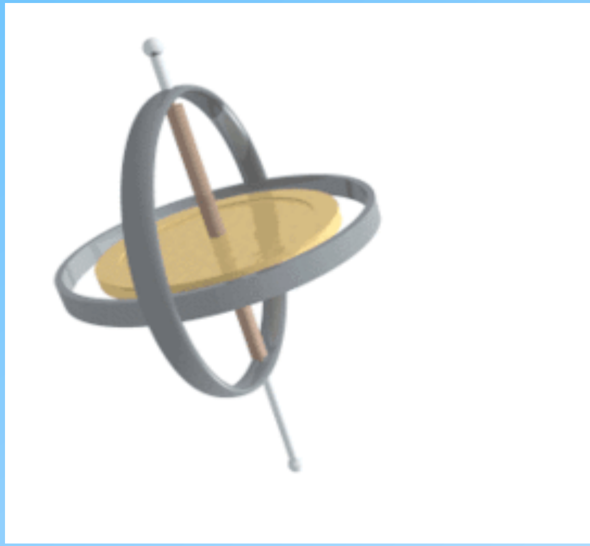
Rotation around two axes



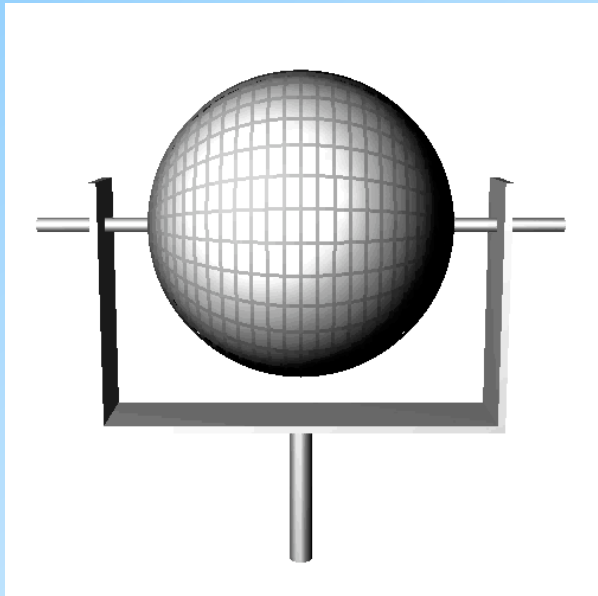
Are these two rotations represented by their sum? No!

Precession

Spin axis itself rotates around another axis.



Precessing motion of a gyroscope



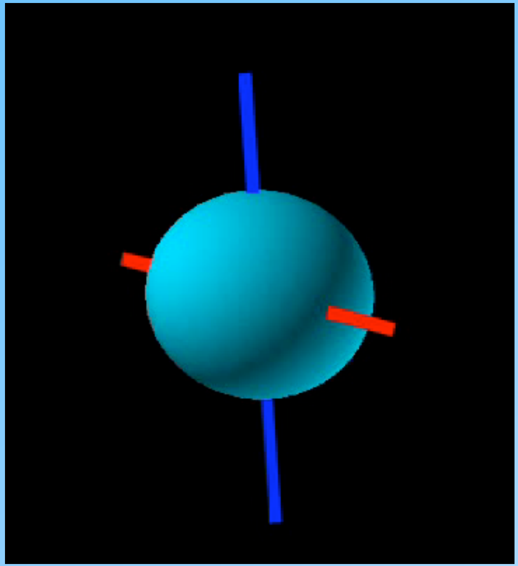
Ω_x

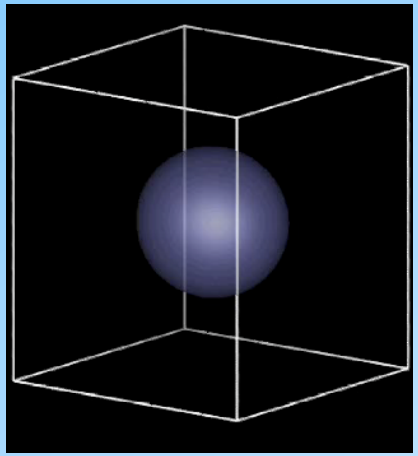
Ω_z

We consider the case where the spinning and precessing rotational axes are perpendicular to each other. Hence, the two rotations do not commute, and thus cannot be represented by their sum.

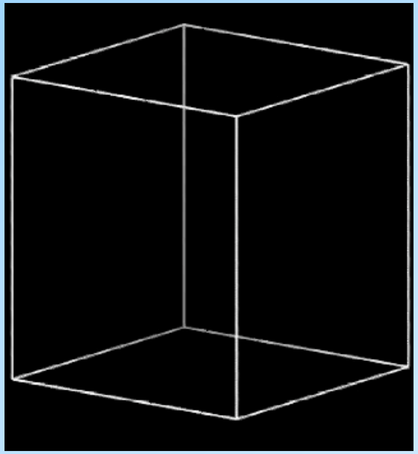
M. Kobayashi and M. Tsubota, Phys. Rev. A76, 045603 (2007)

Two precessions ($\omega_x \times \omega_z$)

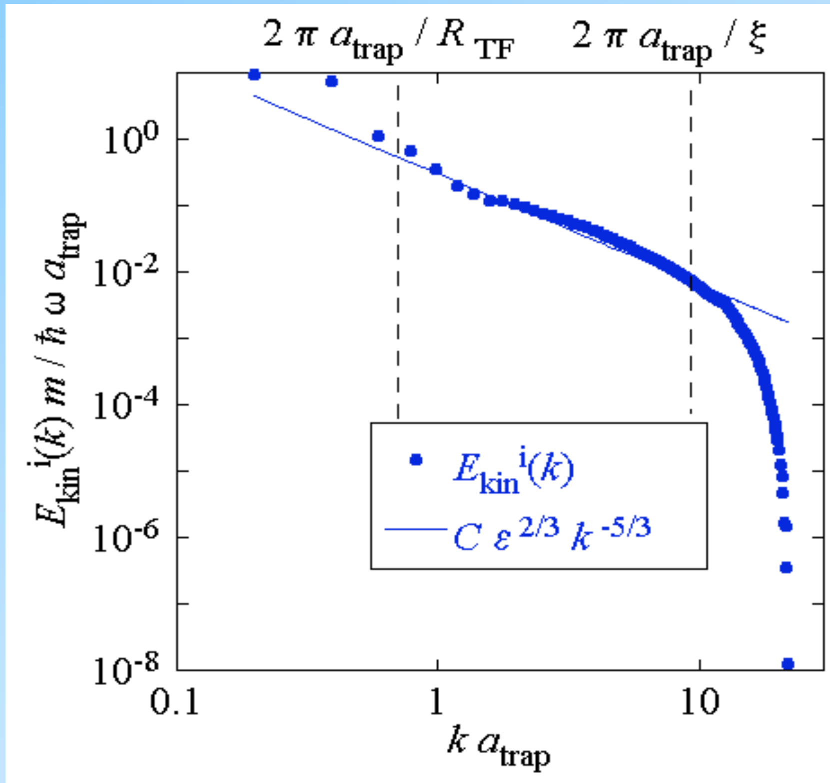




Condensate density



Quantized vortices

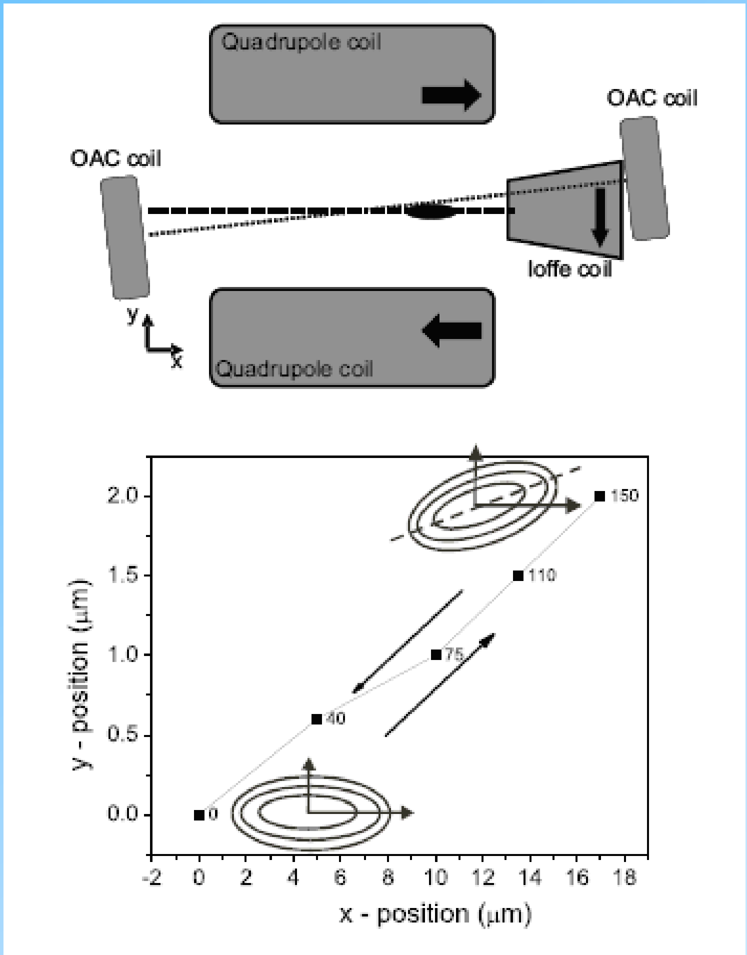


$$n \approx 1.78 \pm 0.194$$

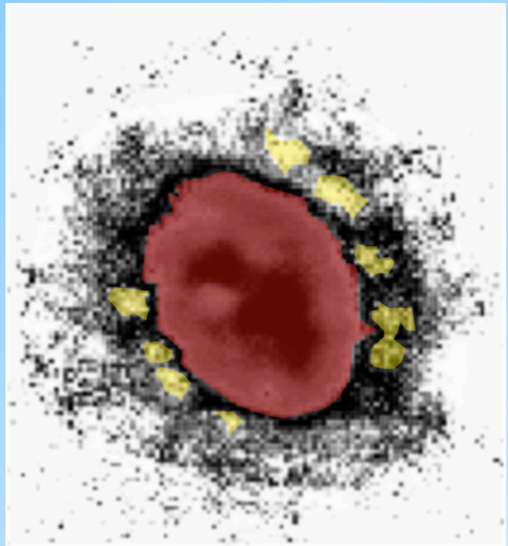
We confirmed a scaling law of the energy spectrum similar to K41.

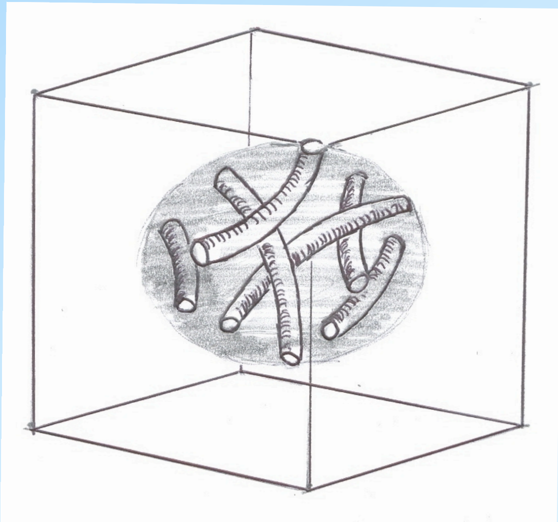
Recently quantum turbulence was realized also in atomic BECs!

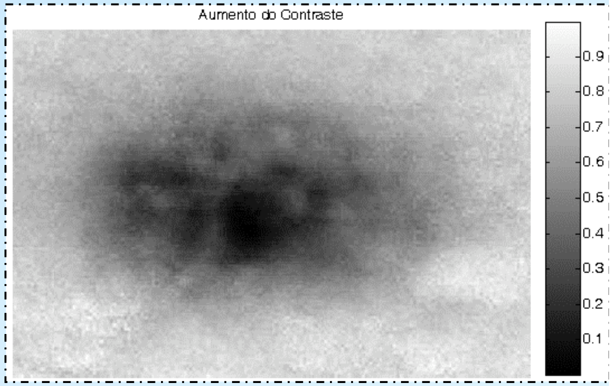
Henn *et al.*, PRL103, 045301(2009)

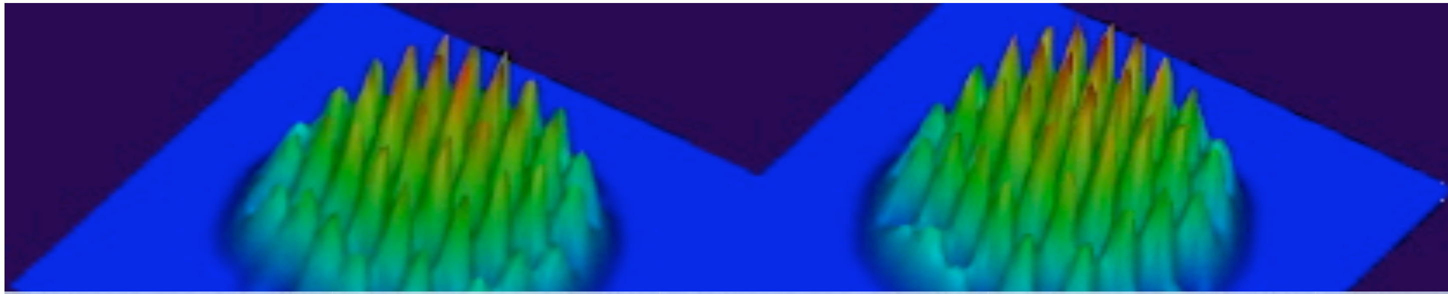


Coupled large amplitude oscillation



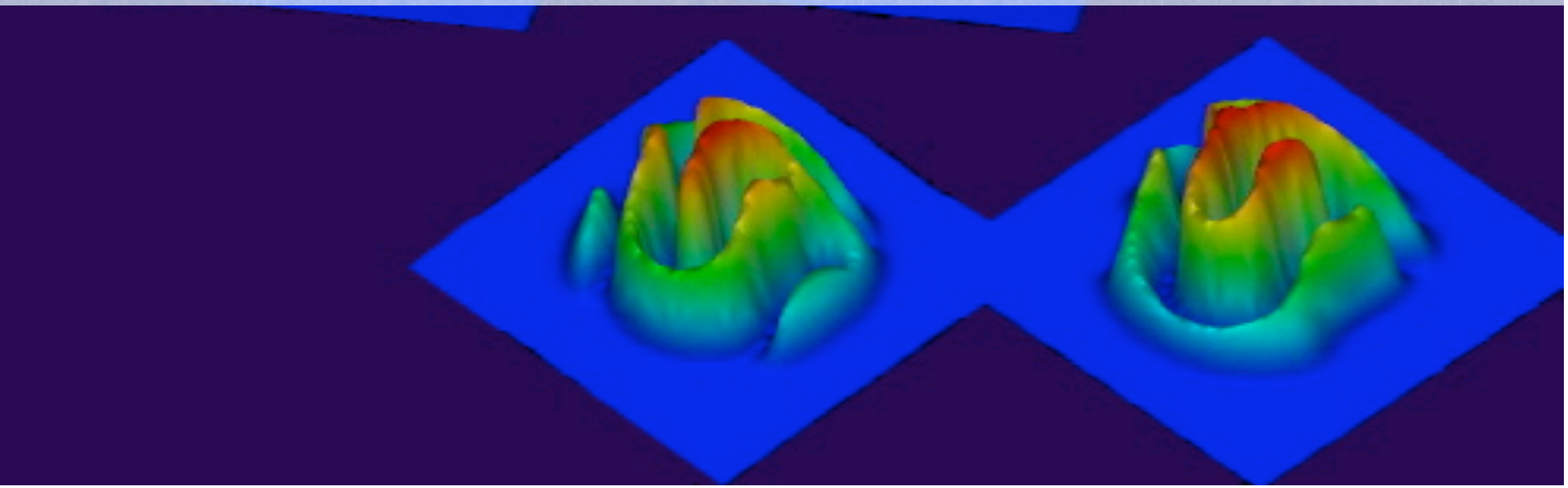






4. Quantized vortices in two-component BECs

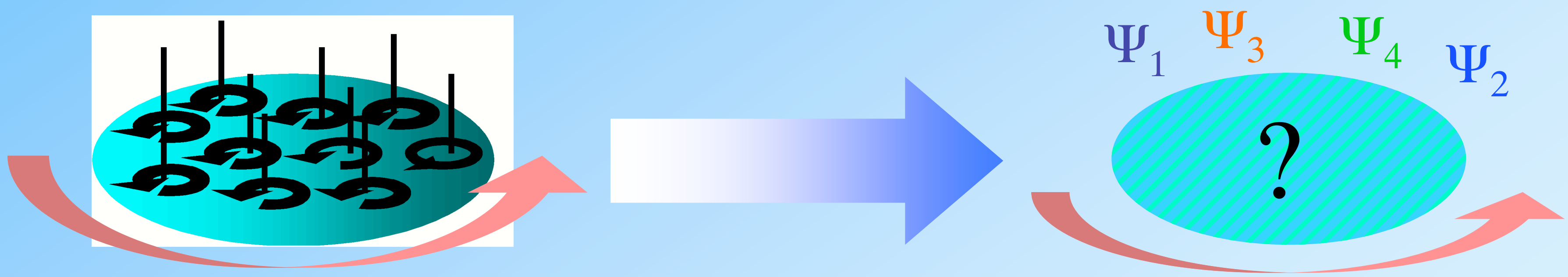
Review article*: *K.Kasamatsu, MT, M.Ueda, Int. J. Mod. Phys. 11, 1835(2005)



Vortices and hydrodynamics in multi-component BECs

Depending on the symmetry, multi-component order parameters can yield various kinds of topological defects.

superfluid ^{3}He, superconductivity with non-s-wave symmetry (Sr_2RuO_4, UPt_3), bilayer quantum Hall systen, nonlinear optics, nuclear physics, cosmology(Neutron star), …

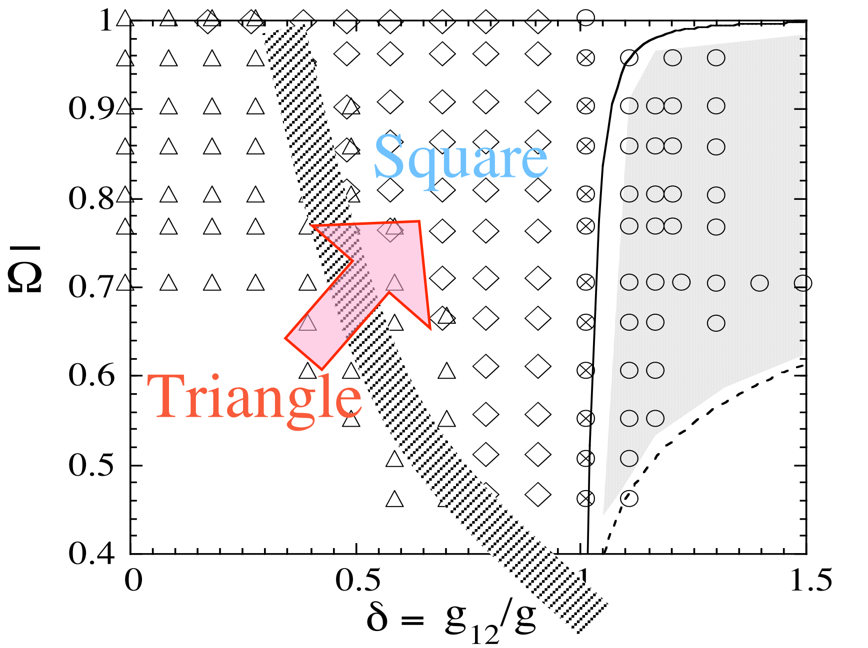


Ω Triangular (Abrikosv)lattice

Ω

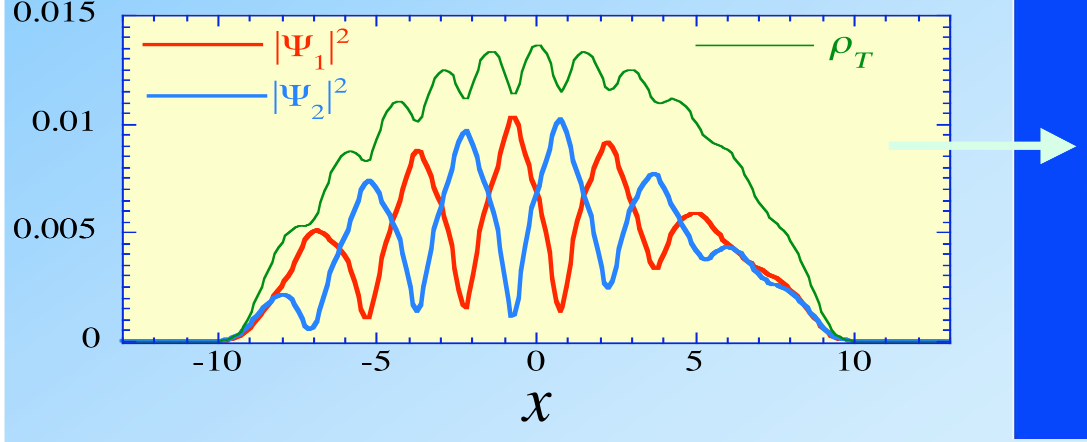
Topological defects in two-component BECs

Vortex lattices in rotating two-component BECs

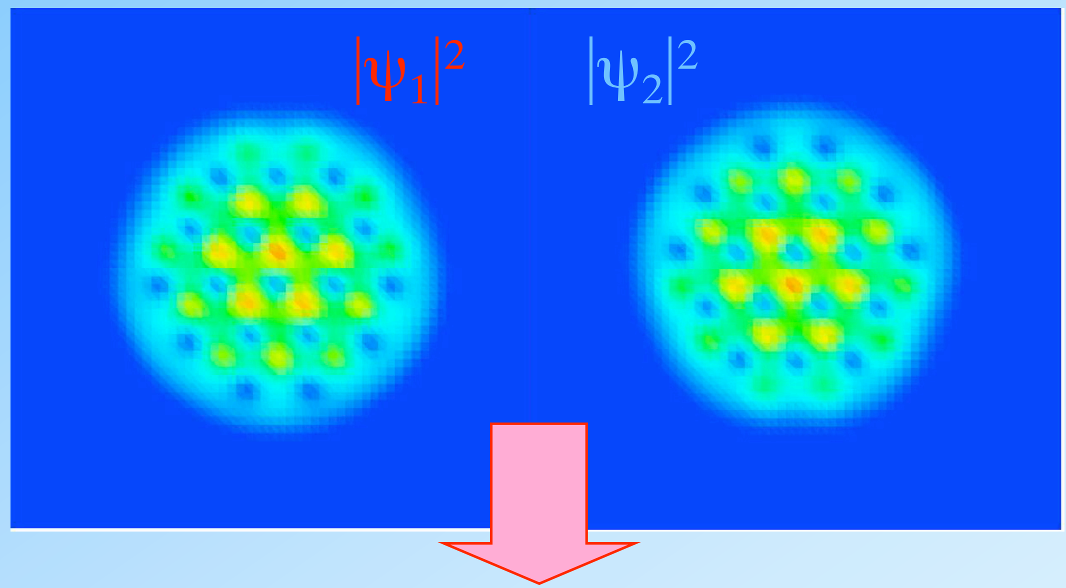


K. Kasamatsu, MT, M. Ueda, PRL91, 150406 (2003)

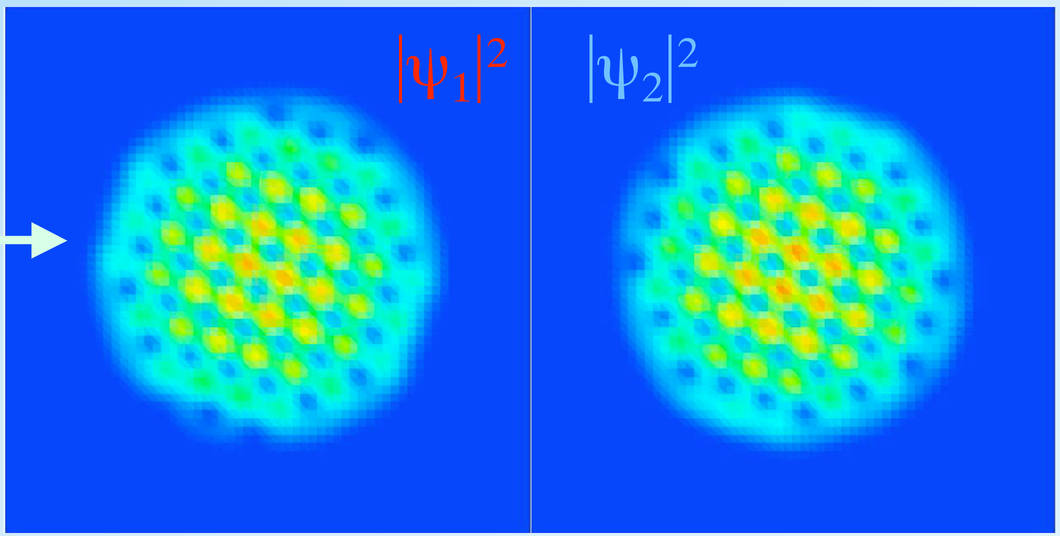
cross section



Triangular lattices



Square lattices



Hydrodynamic instability in two-component BECs

Quantum Kelvin-Helmholtz instability (KHI)

H. Takeuchi, N. Suzuki, K. Kasamatsu, H. Saito, MT, PRB81, 094517 (2010)

Crossover between KHI and counterflow instability

N. Suzuki, H. Takeuchi, K. Kasamatsu, MT, H. Saito, PRA81, 063604 (2010)

Counterflow instability and QT
H. Takeuchi, S. Ishino, MT, PRL105, 205301(2010);
S. Ishino, H. Takeuchi, MT, PRA83, 063602(2011)

Rayleigh-Taylor instability

K. Sasaki, N, Suzuki, D. Akamatsu, H. Saito, PRA80, 042704 (2009)

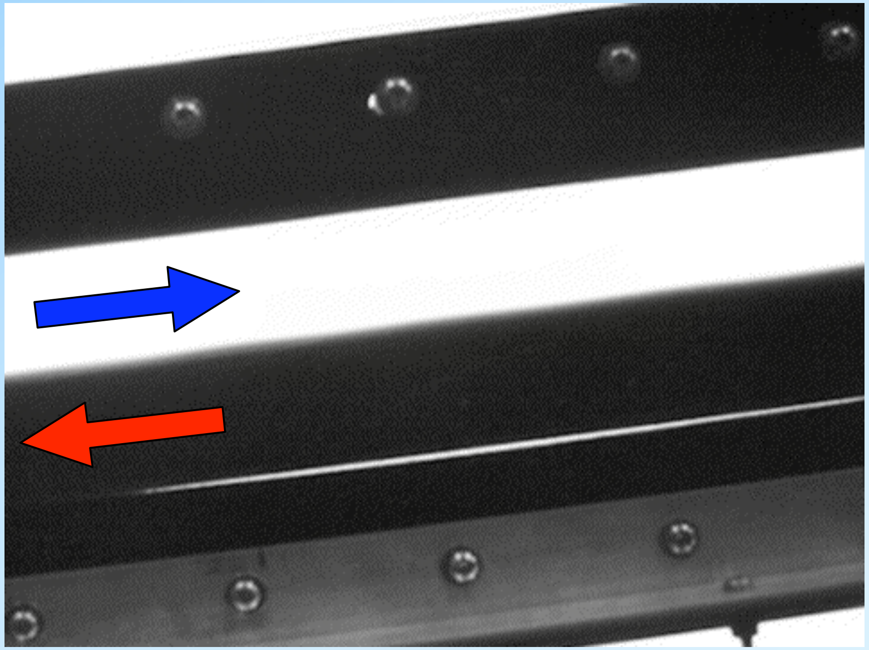
4-1. Quantum Kelvin-Helmholtz instability in two-component BECs

H. Takeuchi, N. Suzuki, K. Kasamatsu, H. Saito, M. Tsubota,

Phys. Rev. B81, 094517(2010).

KHI: Hydrodynamic instability of shear flows

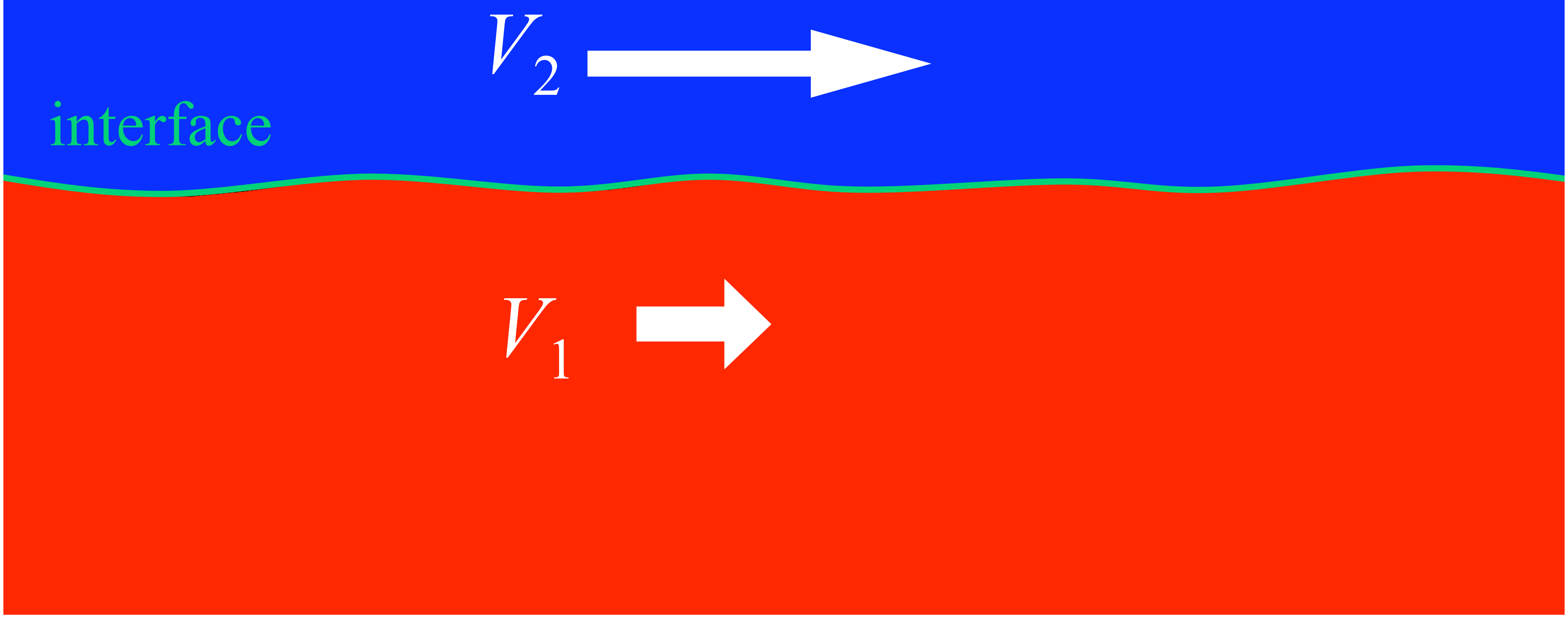
One of the most fundamental instability in classical fluid dynamics



We study the KHI in two-component atomic Bose-Einstein condensates(BECs).

Classical KHI

When the relative velocity $V_d=|V_1-V_2|$ is sufficiently large, the vortex sheet becomes dynamically unstable and the interface modes with complex frequencies are amplified.



Two-component BEC

Two order parameters (macroscopic wave functions) $\Psi_1 \quad \Psi_2$

Coupled Gross-Pitaevskii(GP) equations

$$i\hbar\partial_t\Psi_1 = \left(-\frac{\hbar^2}{2m_1}\nabla^2 + U_1 + g_{11}|\Psi_1|^2 + g_{12}|\Psi_2|^2\right)\Psi_1$$

$$i\hbar\partial_t\Psi_2 = \left(-\frac{\hbar^2}{2m_2}\nabla^2 + U_2 + g_{12}|\Psi_1|^2 + g_{22}|\Psi_2|^2\right)\Psi_2$$

$g = g_{11} = g_{22} \qquad m = m_1 = m_2$

$$\Psi_j(t, \mathbf{r}) = \sqrt{n_j(t, \mathbf{r})}e^{i\Theta_j(t,\mathbf{r})}$$

particle density   phase

$$\mathbf{v}_j = \frac{\hbar}{m_j}\nabla\Theta_j$$

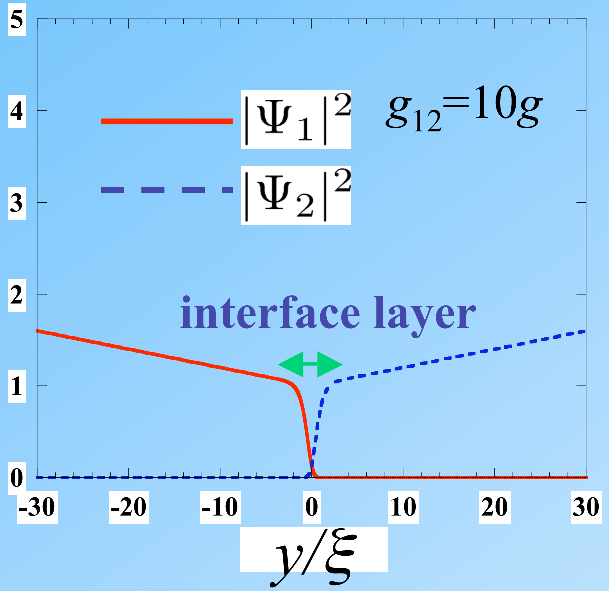
superfluid velocity of component j

Phase-separated two-component BEC

$\Psi_1 \Leftrightarrow \Psi_2$

strong repulsive interaction

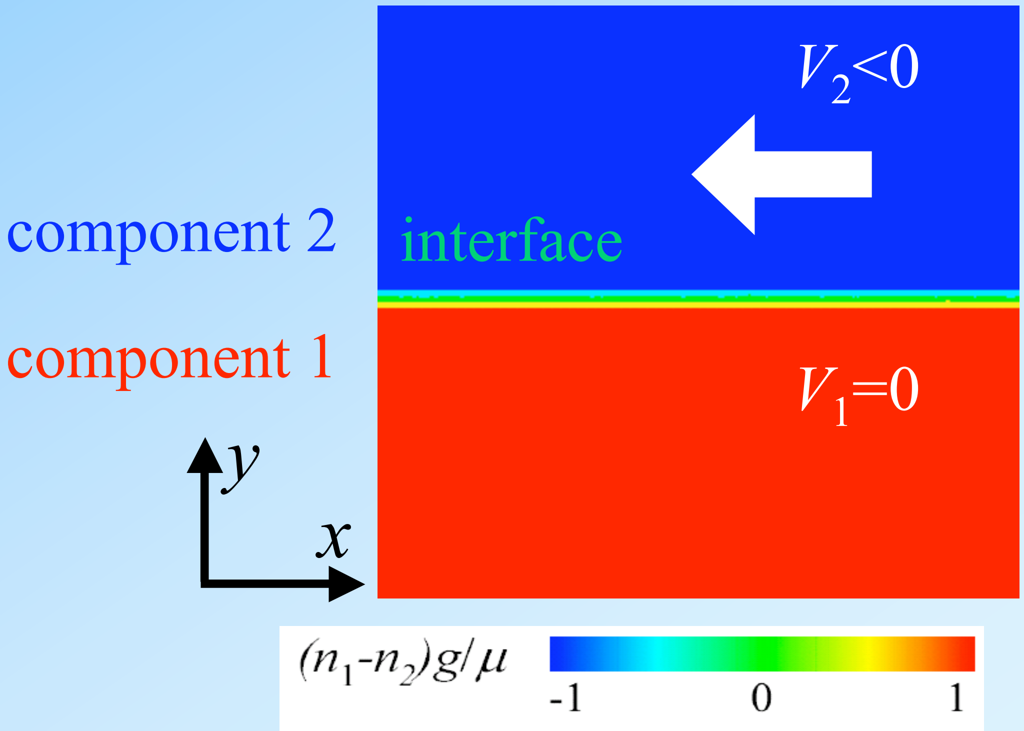
condition for phase separation : $g_{12} > g$



$U_j(y)=f_j y \qquad f_1 = -f_2 > 0$

$\mu=\mu_1 - mV_1^2/2 = \mu_2 - mV_2^2/2 > 0$

$\xi = \sqrt{\hbar^2/(m\mu)}$



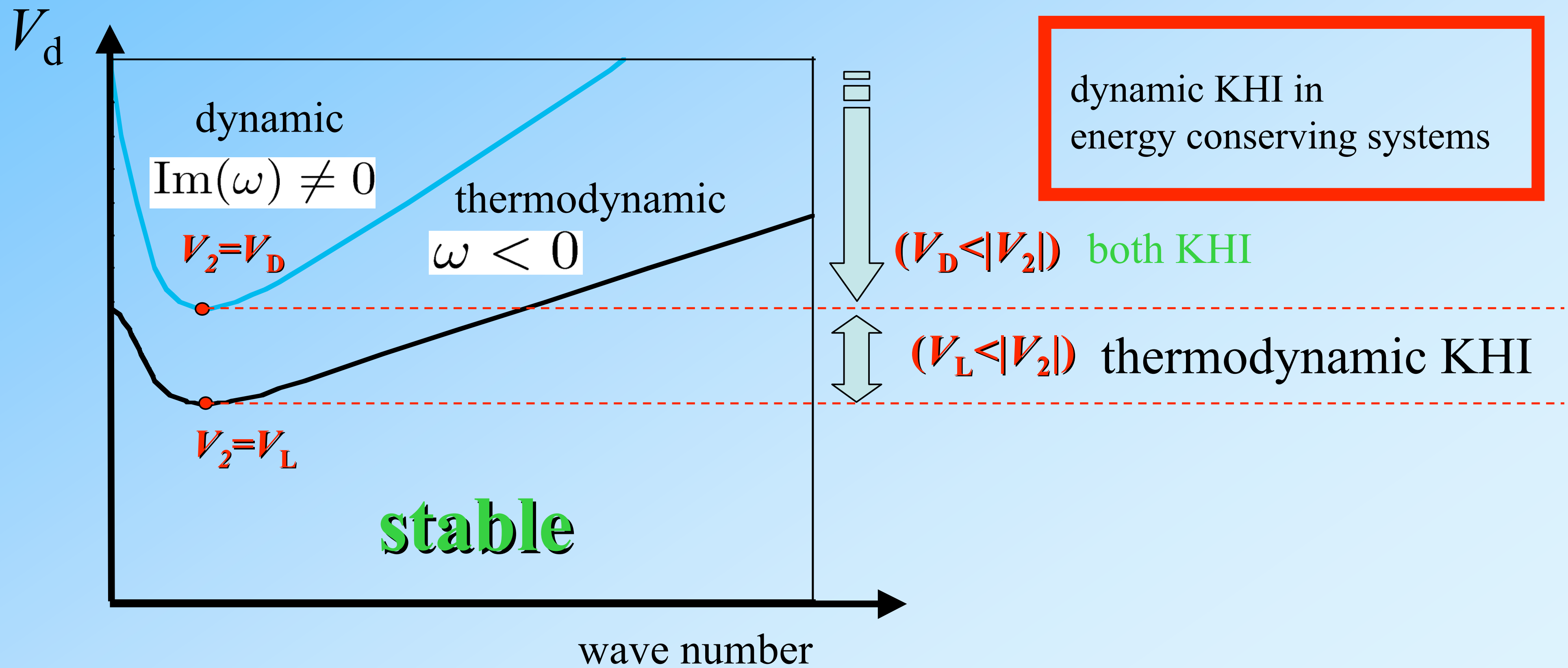
Phase diagram of quantum KHI

-Analysis by the Bogoliubov-de Gennes equations-

dynamic instability $\mathrm{Im}(\omega) \neq 0$ → dynamic KHI (analogue of the classical KHI)

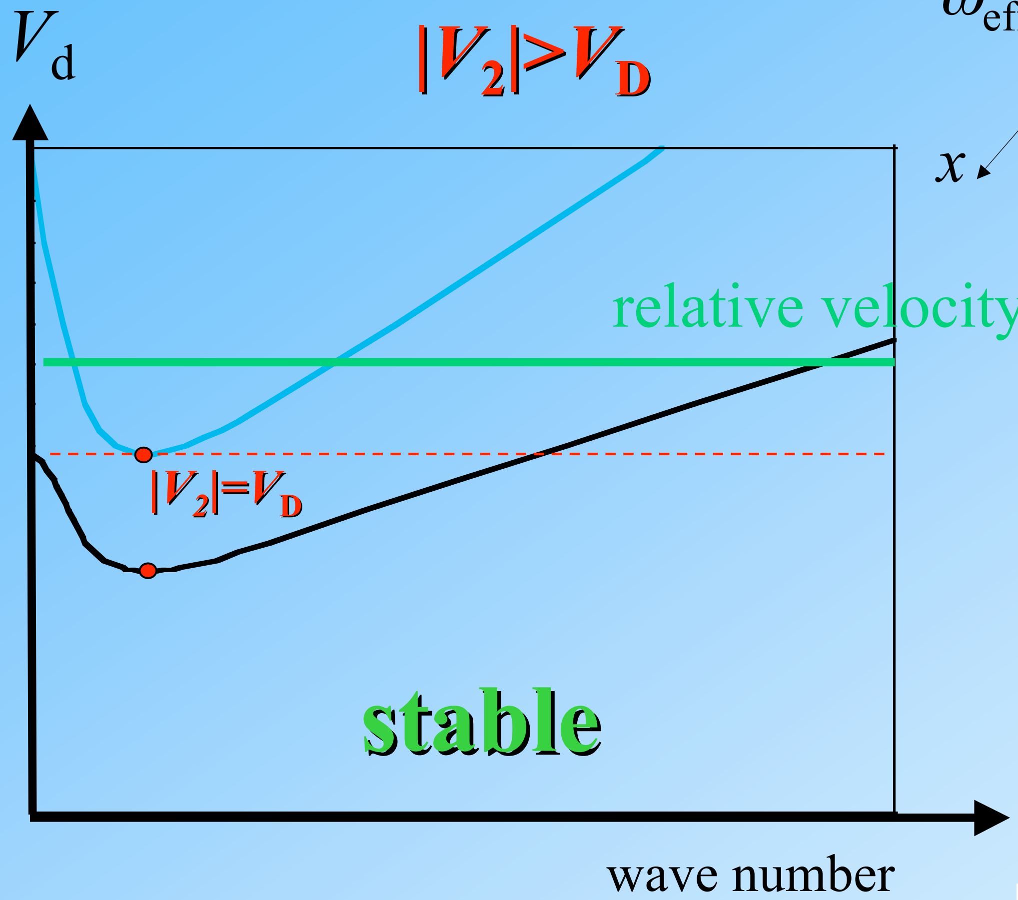
superflow instability $\omega < 0$ → thermodynamic KHI (unique to quantum KHI)

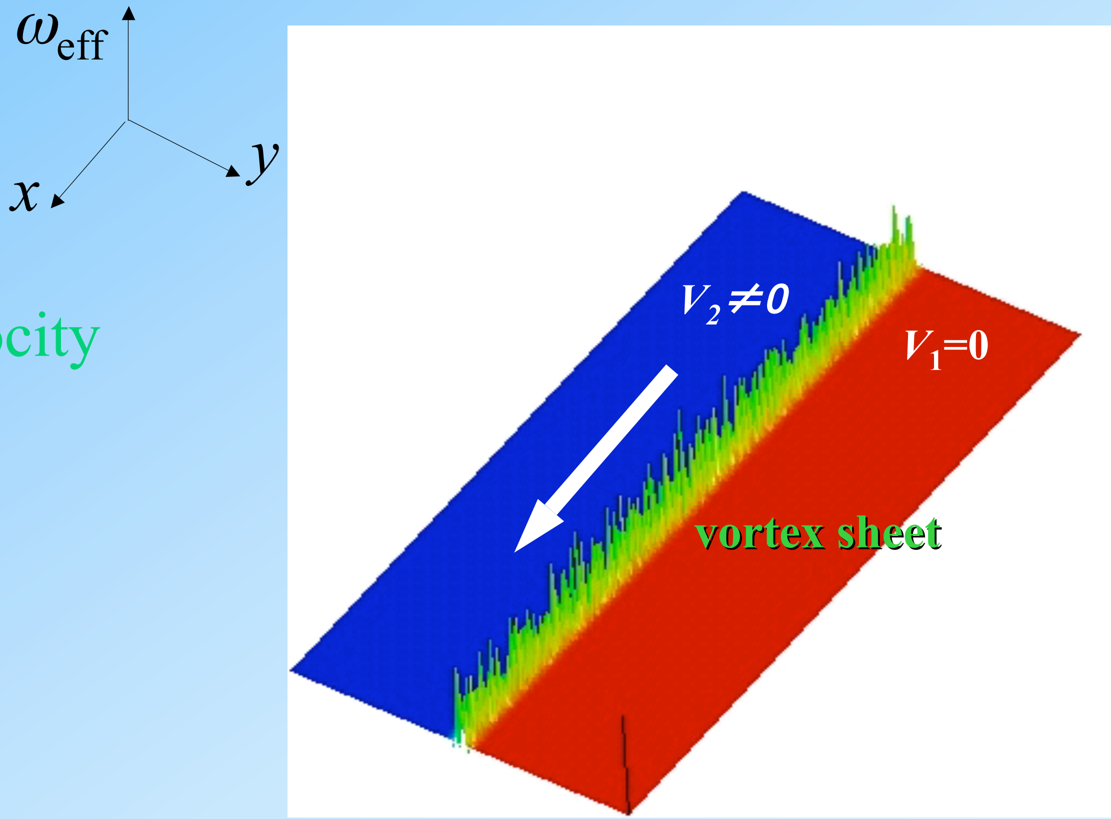
ω: frequency of ripplon



dynamic KHI in
energy conserving systems

Dynamic KHI when the energy is conserved





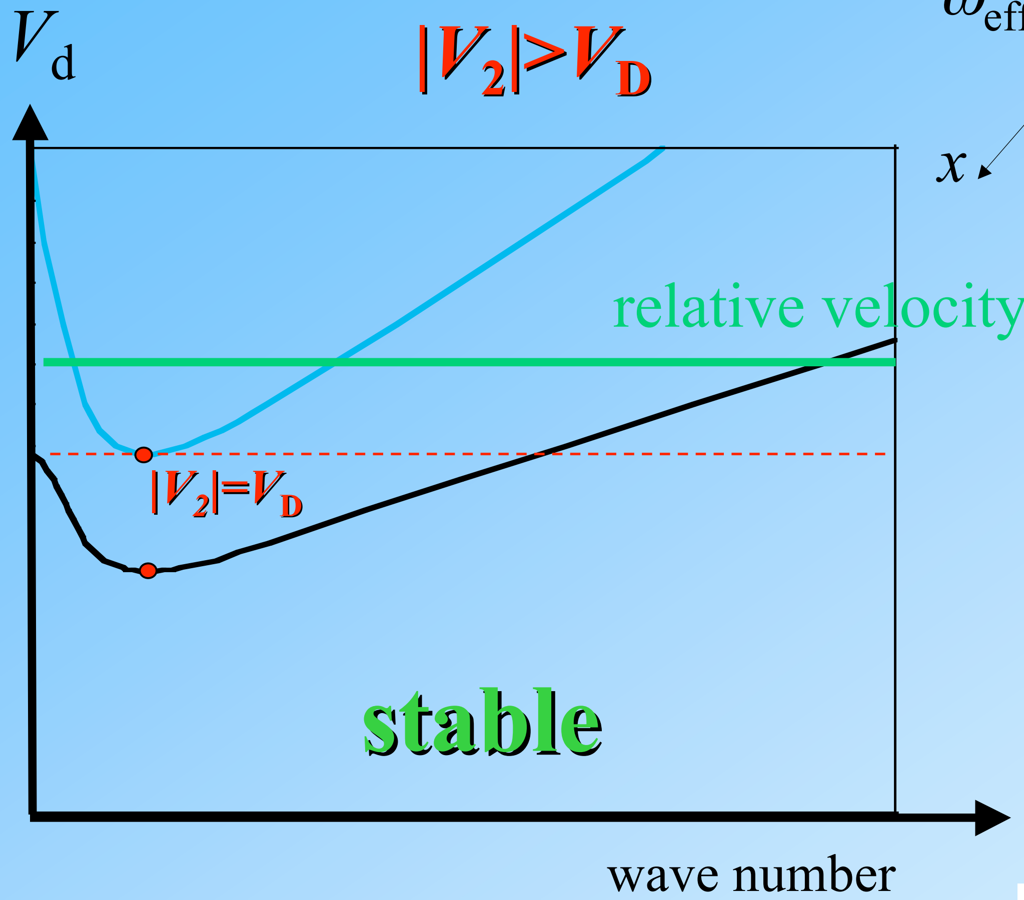
$$\omega_{\rm eff} = (\nabla \times \mathbf{v}_{\rm eff})_z$$

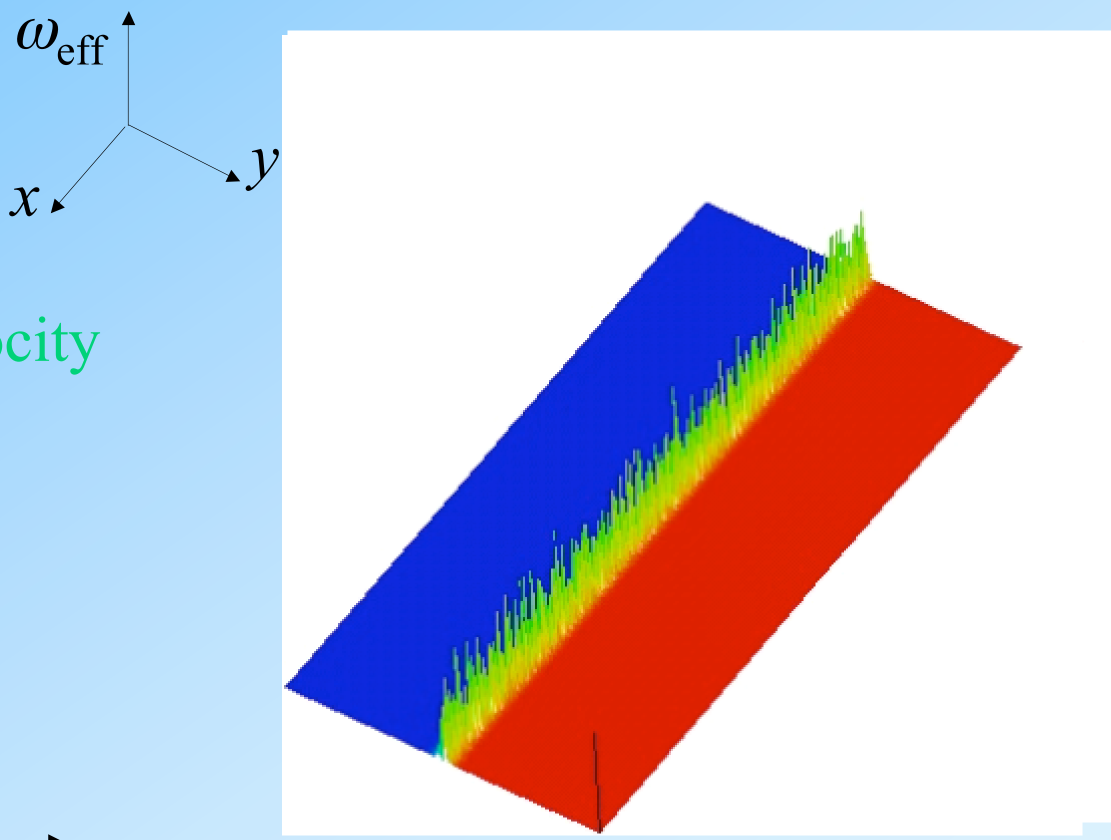
$$\boldsymbol{v}_{\rm eff} = \frac{\boldsymbol{j}_1 + \boldsymbol{j}_2}{n_1 + n_2}$$

$$\boldsymbol{j}_i = n_i \boldsymbol{v}_i$$

effective super-current velocity

Dynamic KHI when the energy is conserved





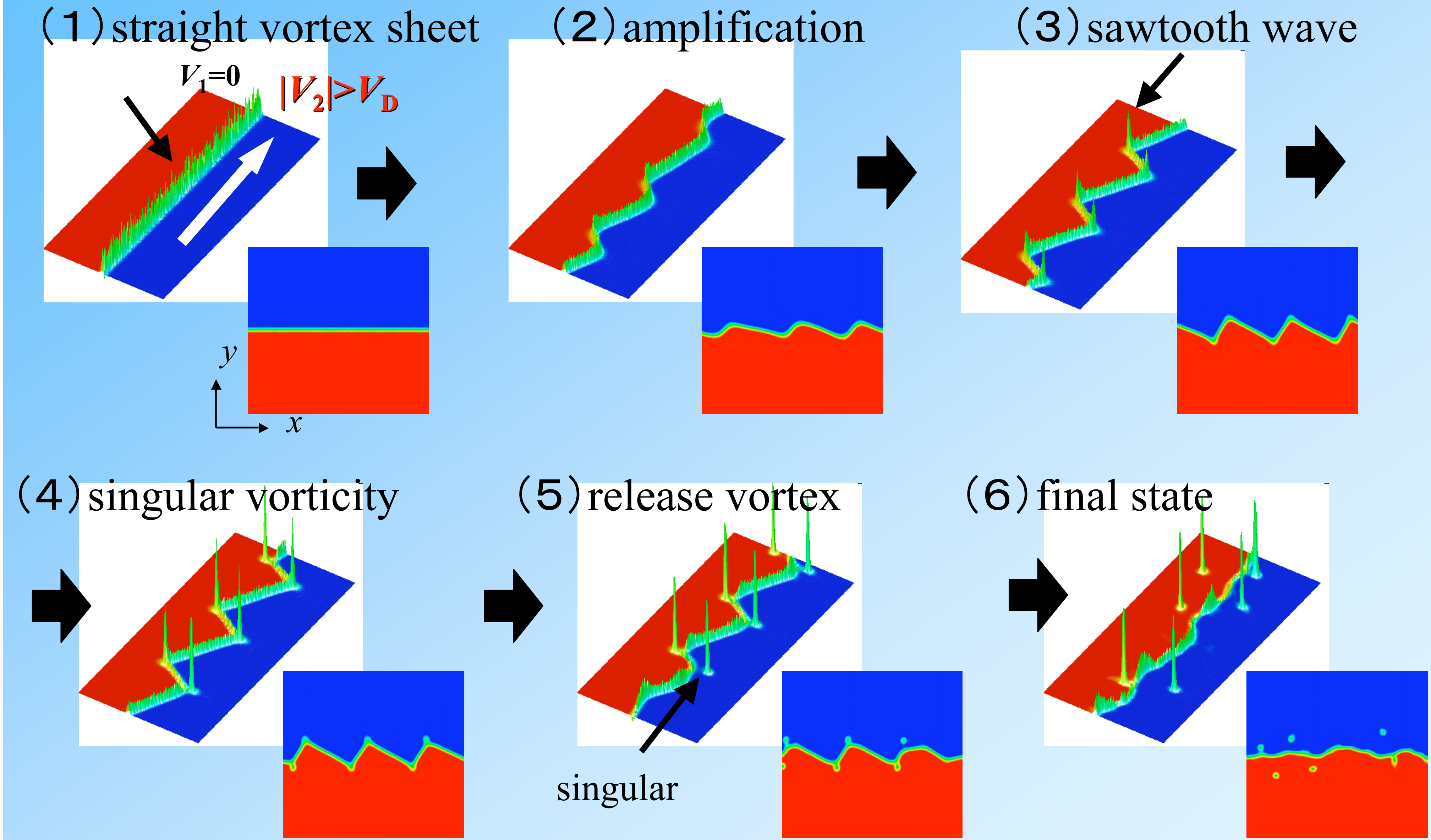
$$\omega_{\mathrm{eff}} = (\nabla \times \mathbf{v}_{\mathrm{eff}})_z$$

$$\boldsymbol{v}_{\mathrm{eff}} = \frac{\boldsymbol{j}_1 + \boldsymbol{j}_2}{n_1 + n_2}$$

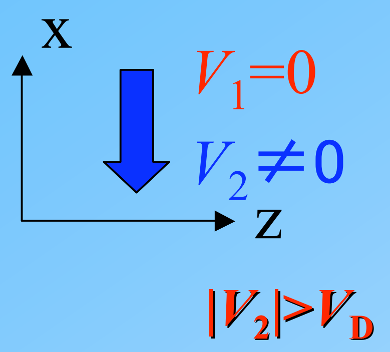
$$\boldsymbol{j}_i = n_i \boldsymbol{v}_i$$

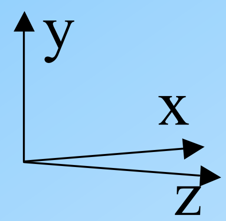
effective super-current velocity

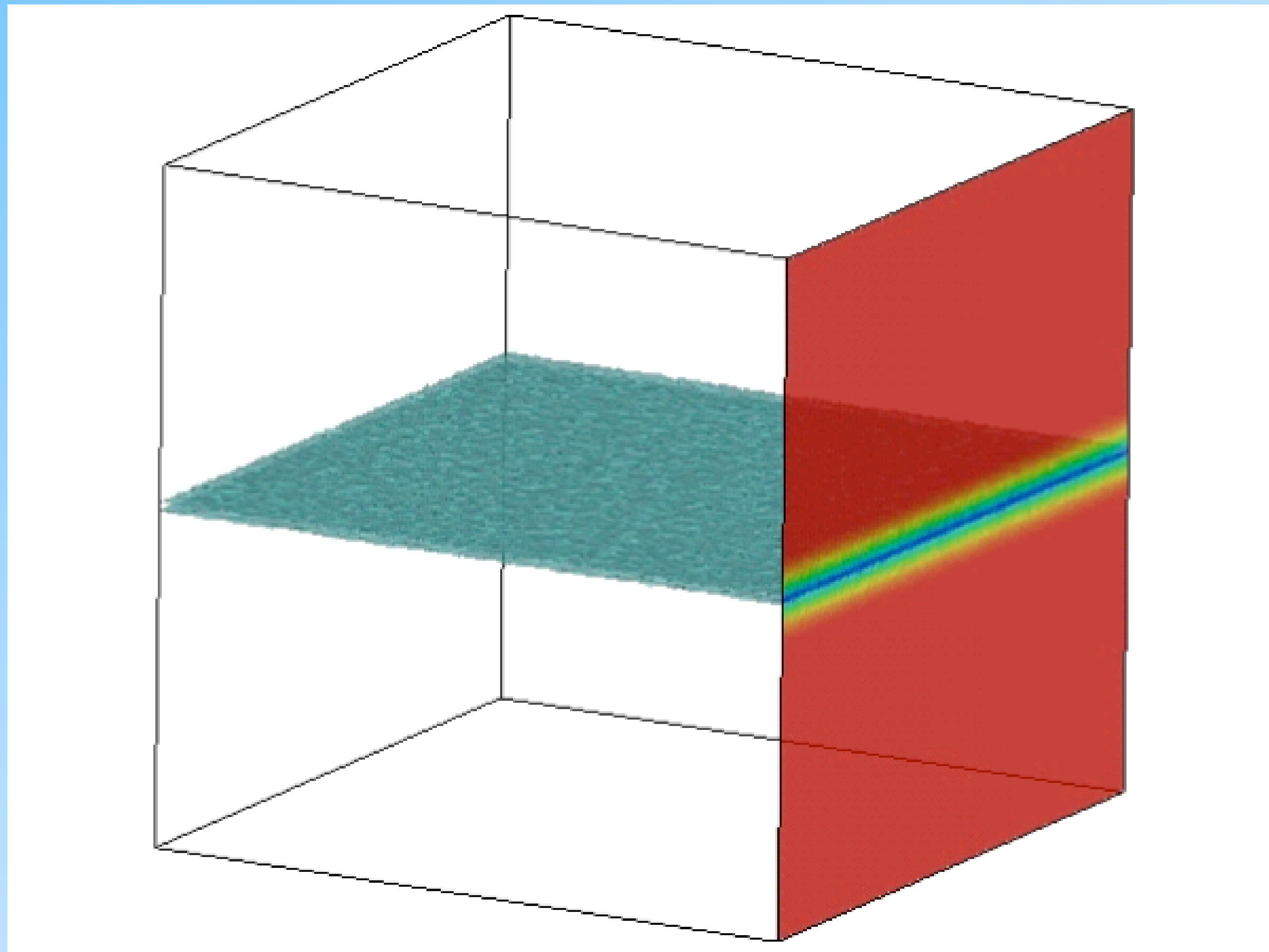
Dynamic KHI when the energy is conserved



Dynamic KHI in 3D system





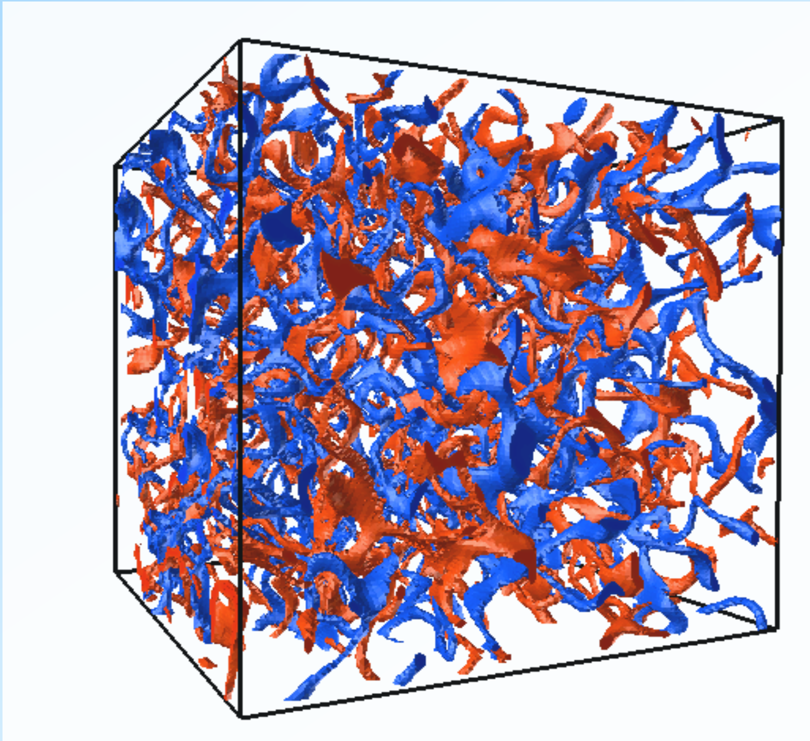


Kelvin waves cause more complicated dynamics towards quantum turbulence.

4-2. Counterflow of two-component BECs: two-component quantum turbulence

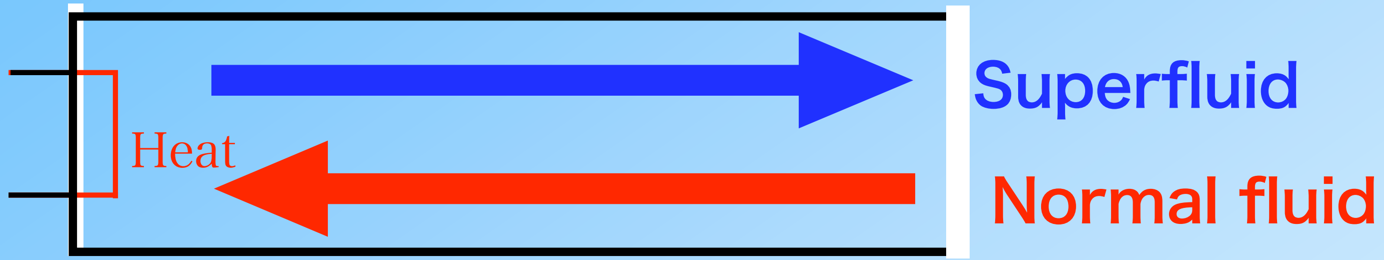
H. Takeuchi, S. Ishino, MT, Phys. Rev. Lett.105, 205301(2010);
S. Ishino, H. Takeuchi, MT, Phys. Rev. A83, 063602(2011)

Multi-component BECs are rich stages for exotic instability and vortex structures!



Motivation

Thermal counter flow



When the relative velocity exceeds some critical value, the superfluid becomes turbulent.

Counterflow of two-component BECs

When the relative velocity exceeds some critical value, two BECs are expected to become unstable and turbulent.

Two-component GP model

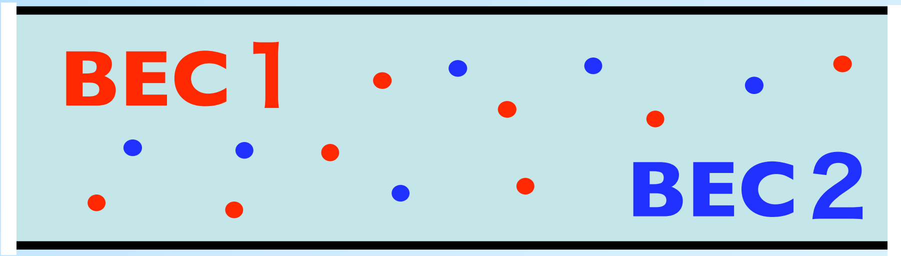
$$i\hbar\frac{\partial}{\partial t}\Psi_1 = \frac{\mathbf{p}_1^2}{2m_1}\Psi_1 + g_{11}|\Psi_1|^2\Psi_1 + g_{12}|\Psi_2|^2\Psi_1$$

$$i\hbar\frac{\partial}{\partial t}\Psi_2 = \frac{\mathbf{p}_2^2}{2m_2}\Psi_2 + g_{22}|\Psi_2|^2\Psi_2 + g_{12}|\Psi_1|^2\Psi_2$$

g_{11}, g_{22} : intracomponent interaction

g_{12} : intercomponent interaction

$g_{11}g_{22} > g_{12}^2$ ➡ The mixture is stable.



Two-component GP model

$$i\hbar\frac{\partial}{\partial t}\Psi_1 = \frac{\mathbf{p}_1^2}{2m_1}\Psi_1 + g_{11}|\Psi_1|^2\Psi_1 + g_{12}|\Psi_2|^2\Psi_1$$

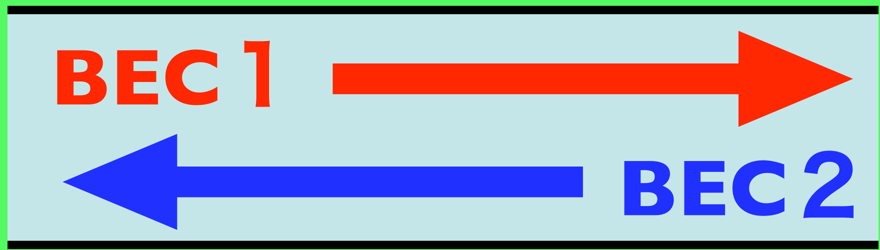
$$i\hbar\frac{\partial}{\partial t}\Psi_2 = \frac{\mathbf{p}_2^2}{2m_2}\Psi_2 + g_{22}|\Psi_2|^2\Psi_2 + g_{12}|\Psi_1|^2\Psi_2$$

g_{11}, g_{22} : intracomponent interaction

g_{12} : intercomponent interaction

$g_{11}g_{22} > g_{12}^2$ ➡ The mixture is stable.

However,



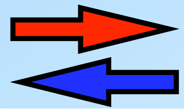
The large relative velocity should make it unstable.

Ref. V. I. Yukalov and E. P. Yukalova, Laser Phys. Lett. **1**, 50 (2004)

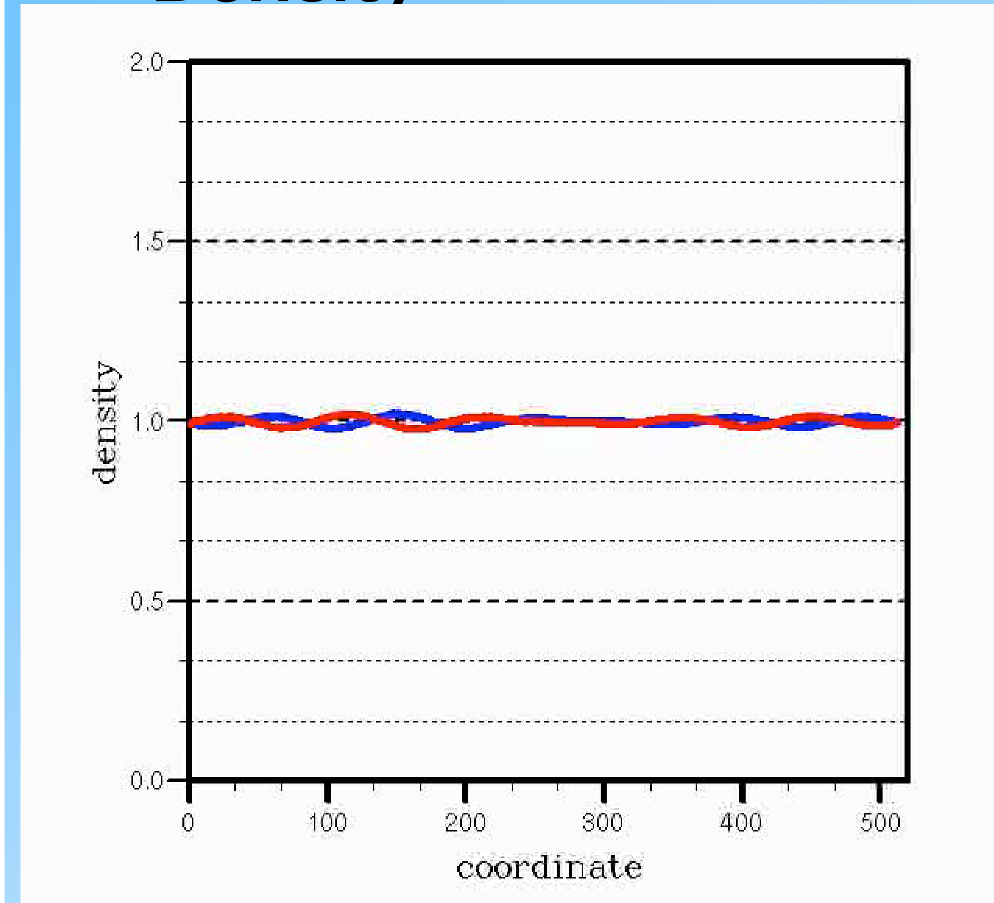
Results (One-dimension)

$n_1 = n_2 = 1.0$, $\mathbf{v}_1 = -\mathbf{v}_2$, $\gamma = 0.5$

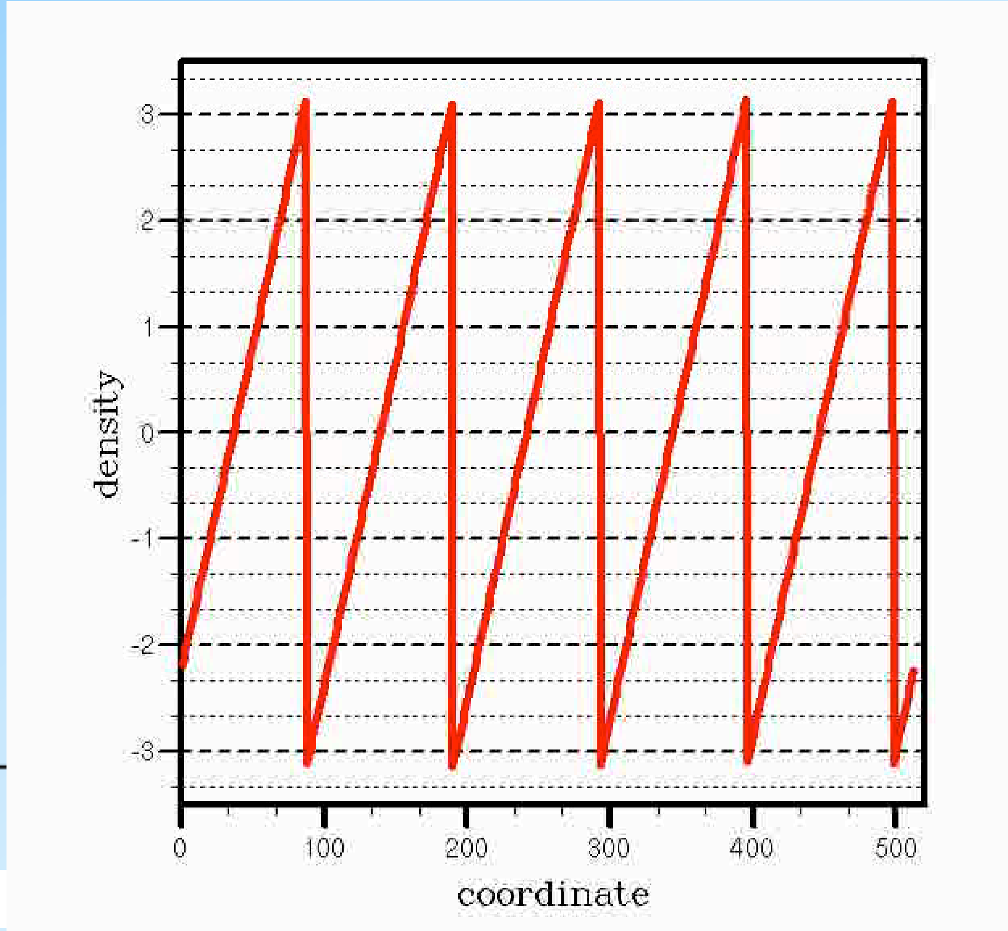
Direction of flow



Density



Phase



Results (One-dimension)

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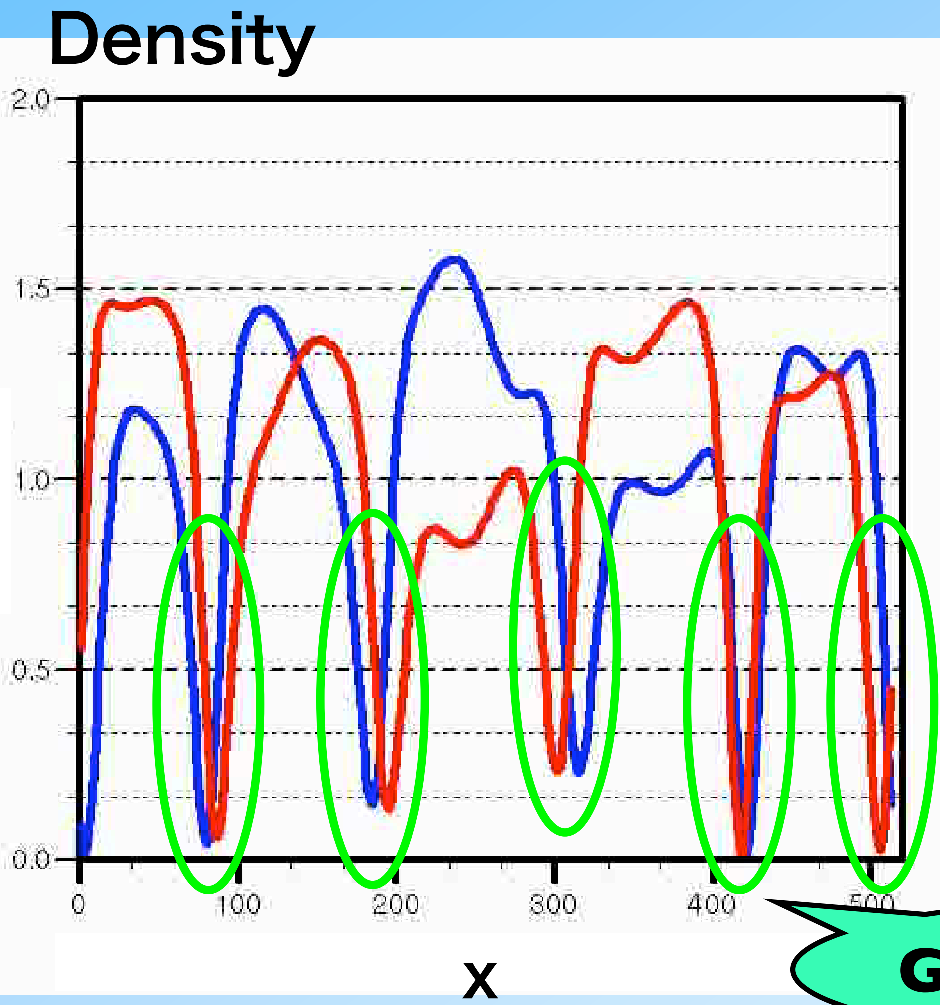
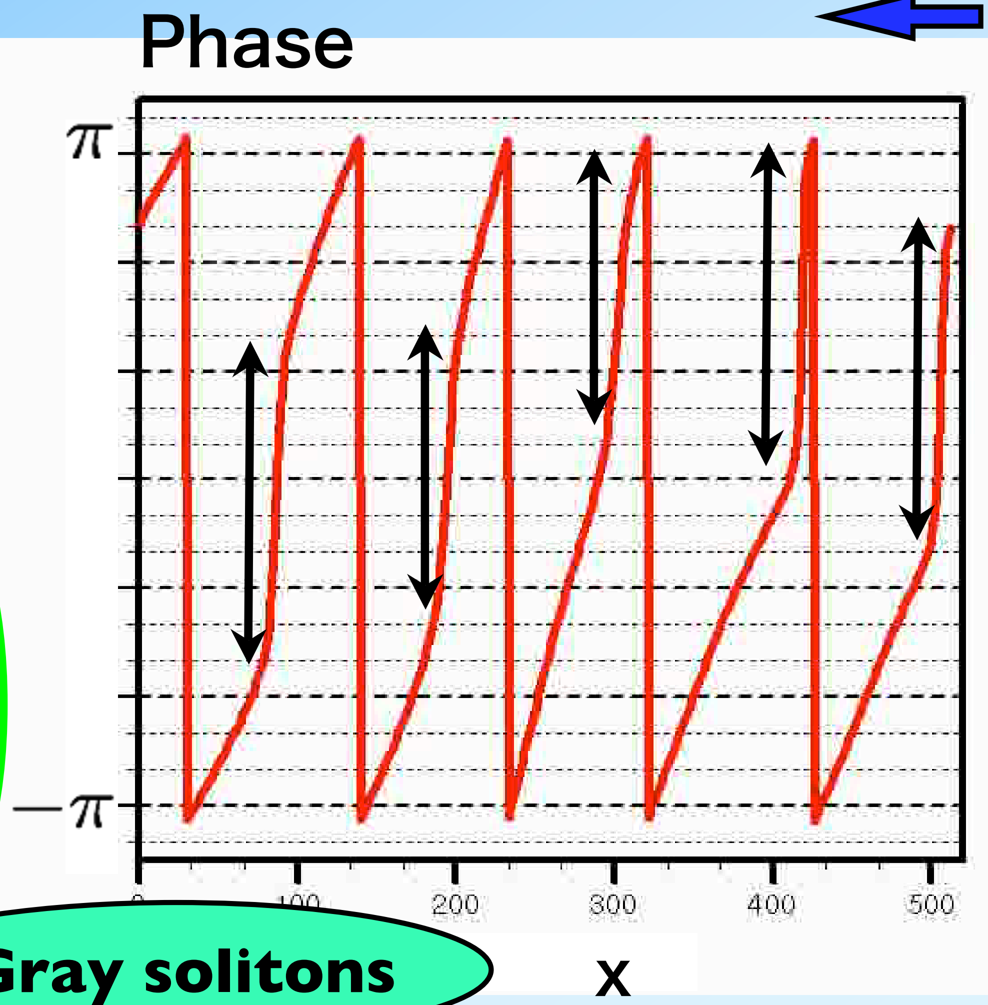
Flow direction



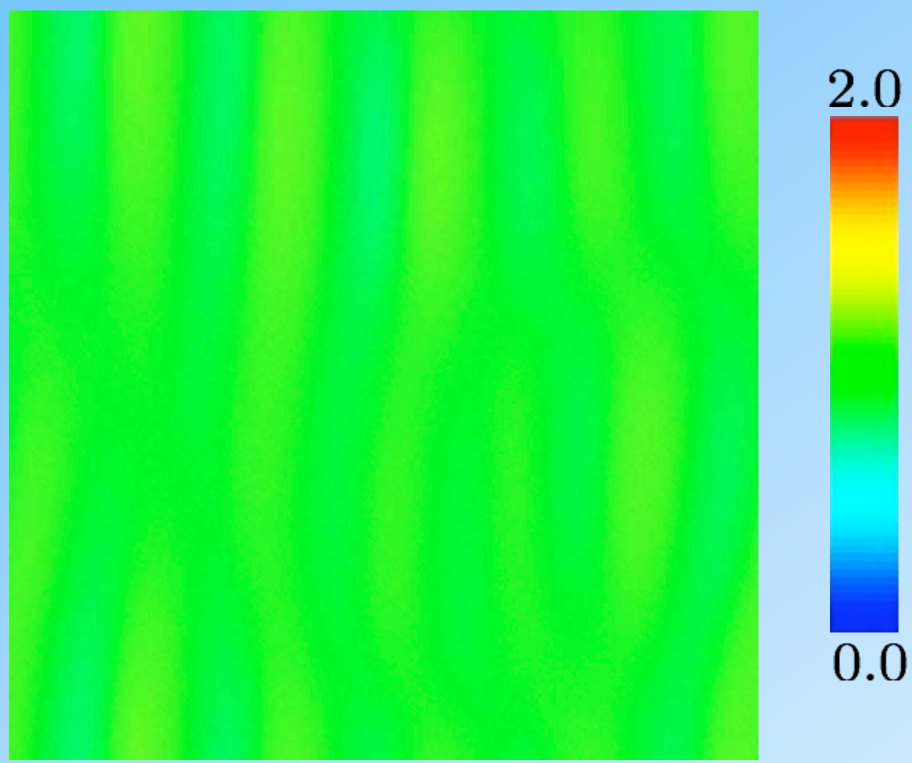


Results (two-dimension)

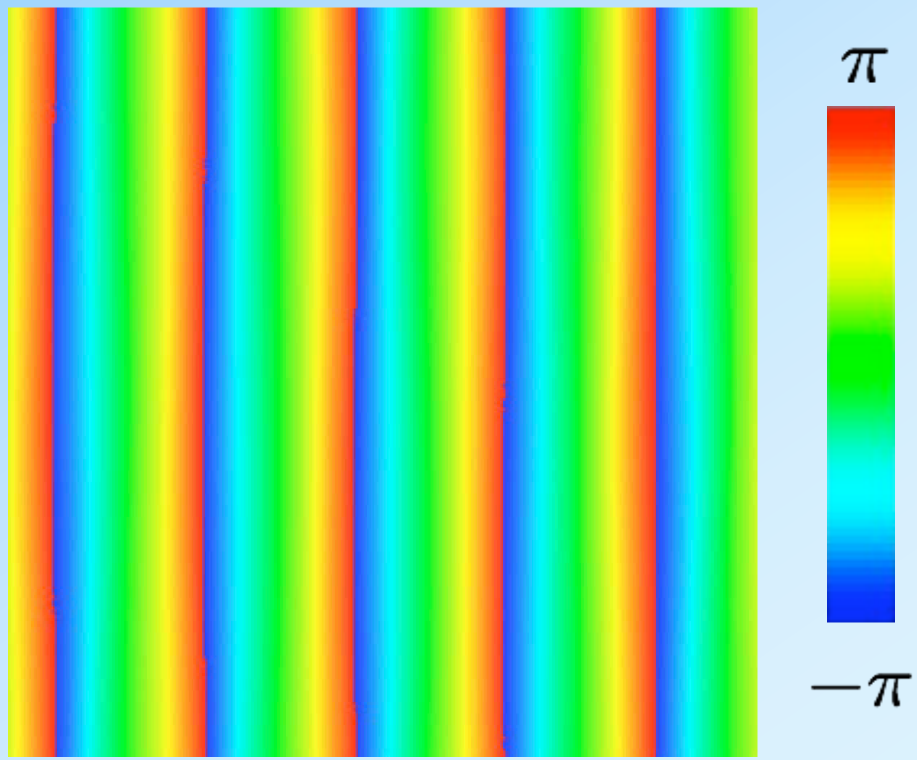
$n_1 = n_2 = 1.0$, $\mathbf{v}_1 = -\mathbf{v}_2$, $\gamma = 0.5$

Flow direction

Density



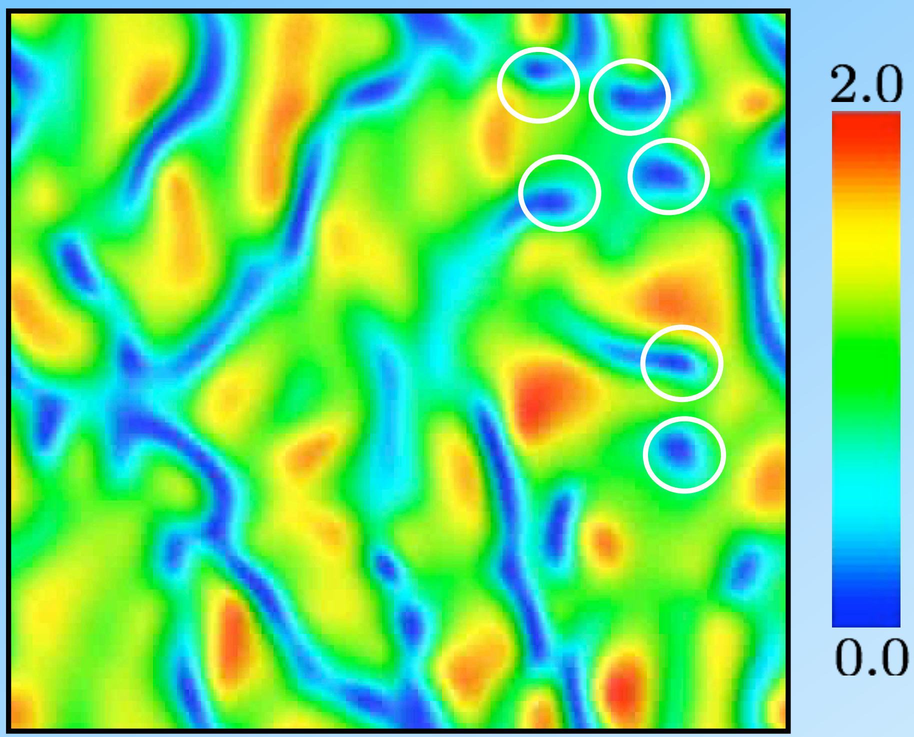
Phase

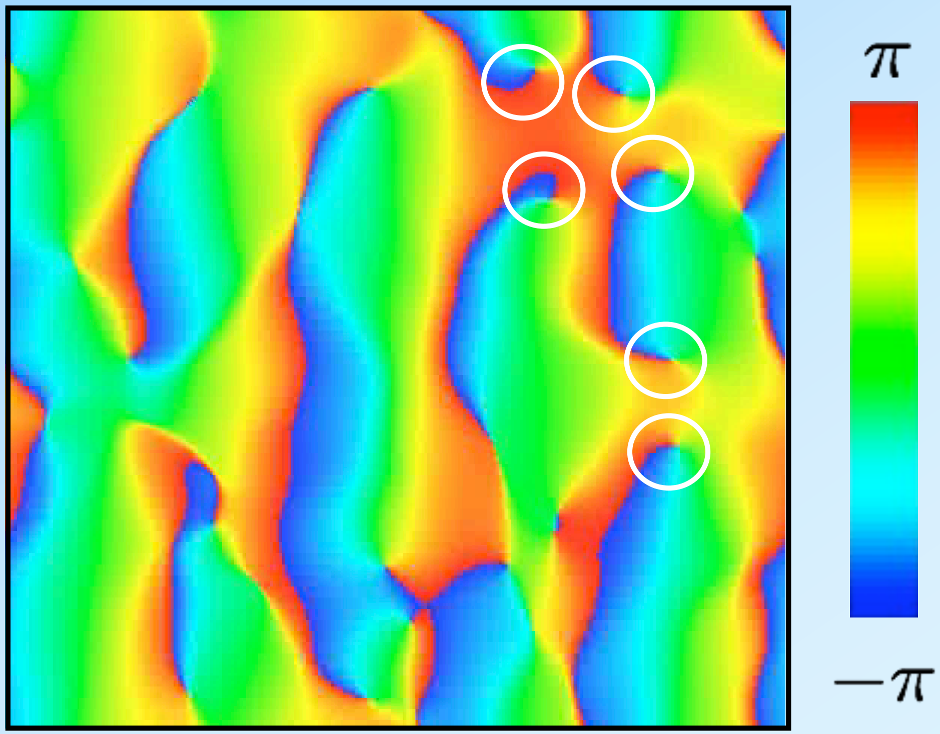


Results (two-dimension)

$n_1 = n_2 = 1.0$, $\mathbf{v}_1 = -\mathbf{v}_2$, $\gamma = 0.5$

Flow direction

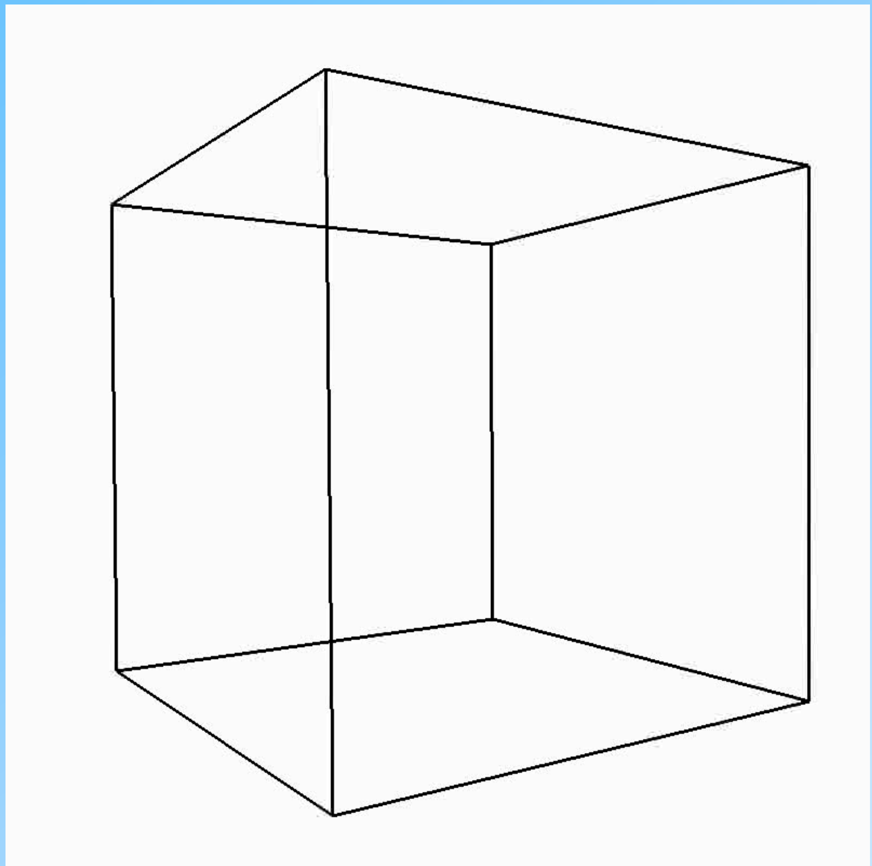


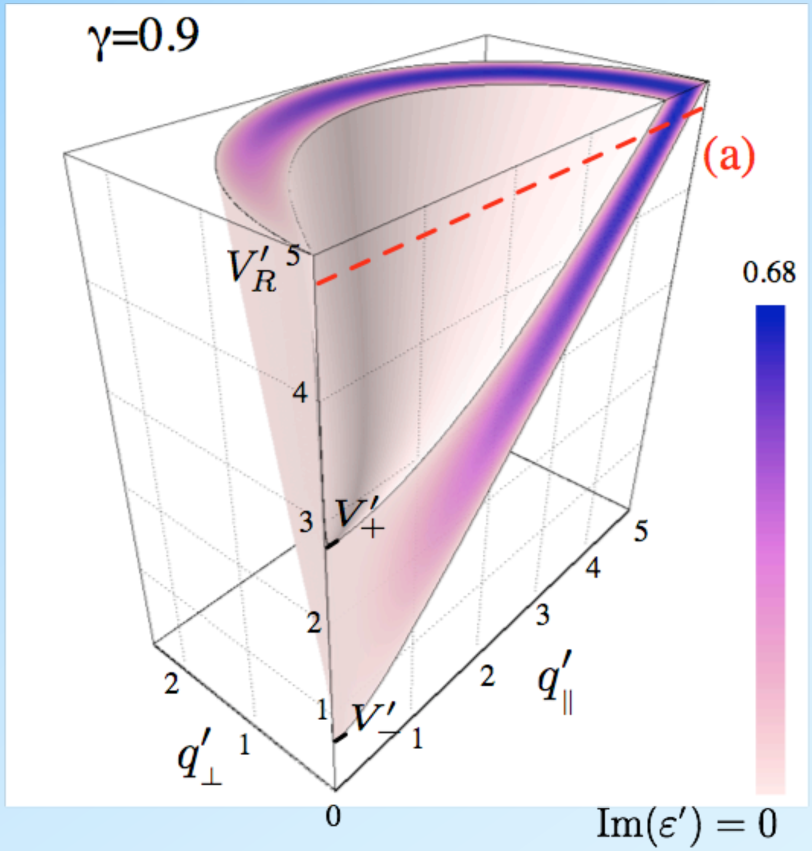


The solitons decay to vortex pairs through snake instability.

3D 2-component QT

$$|\Psi_1|^2/n = |\Psi_2|^2/n = 1, \ \ V_R' = 4.71, \ \ \gamma = 0.9$$



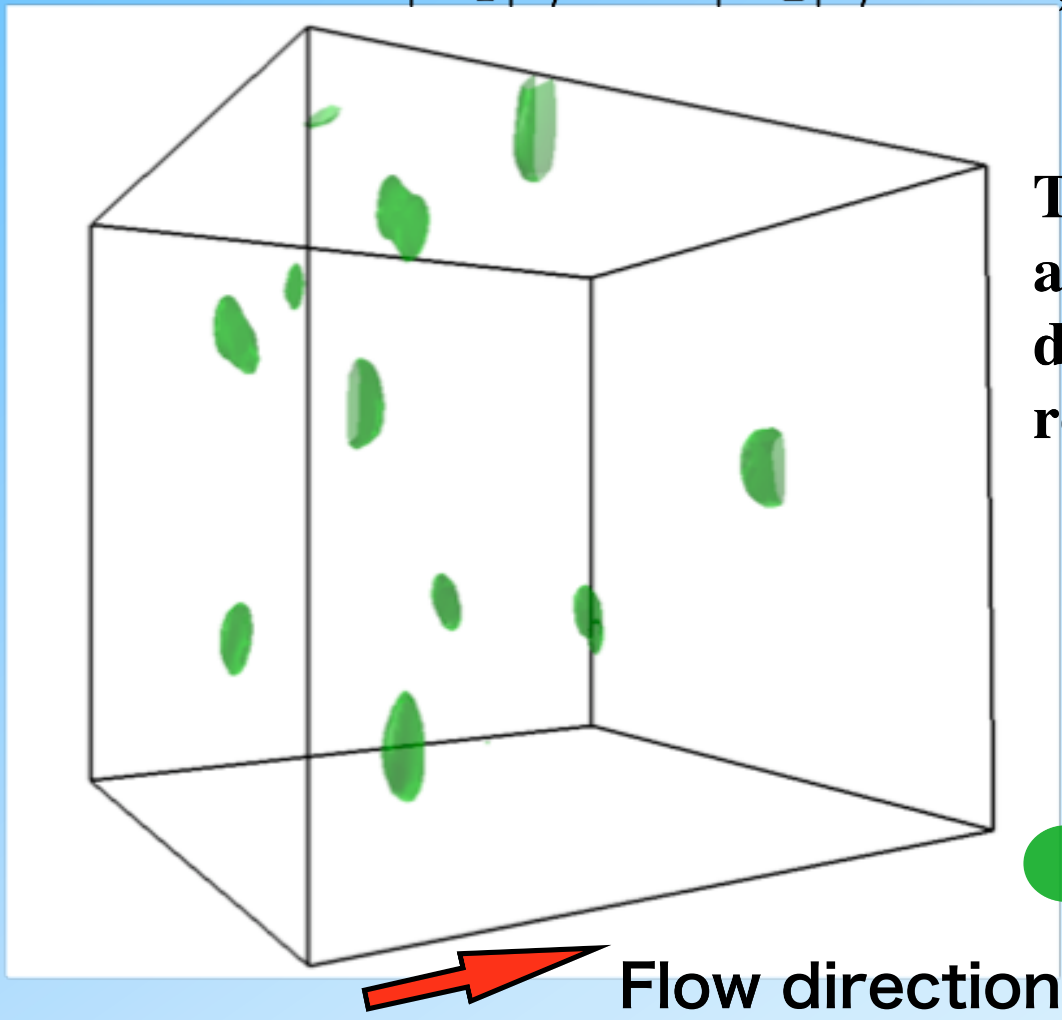


Flow direction

Solitons → Vortex loops → QT

Scenario to turbulence (1)

$$|\Psi_1|^2/n = |\Psi_2|^2/n = 1, \;\; V_R' = 4.71, \;\; \gamma = 0.9$$

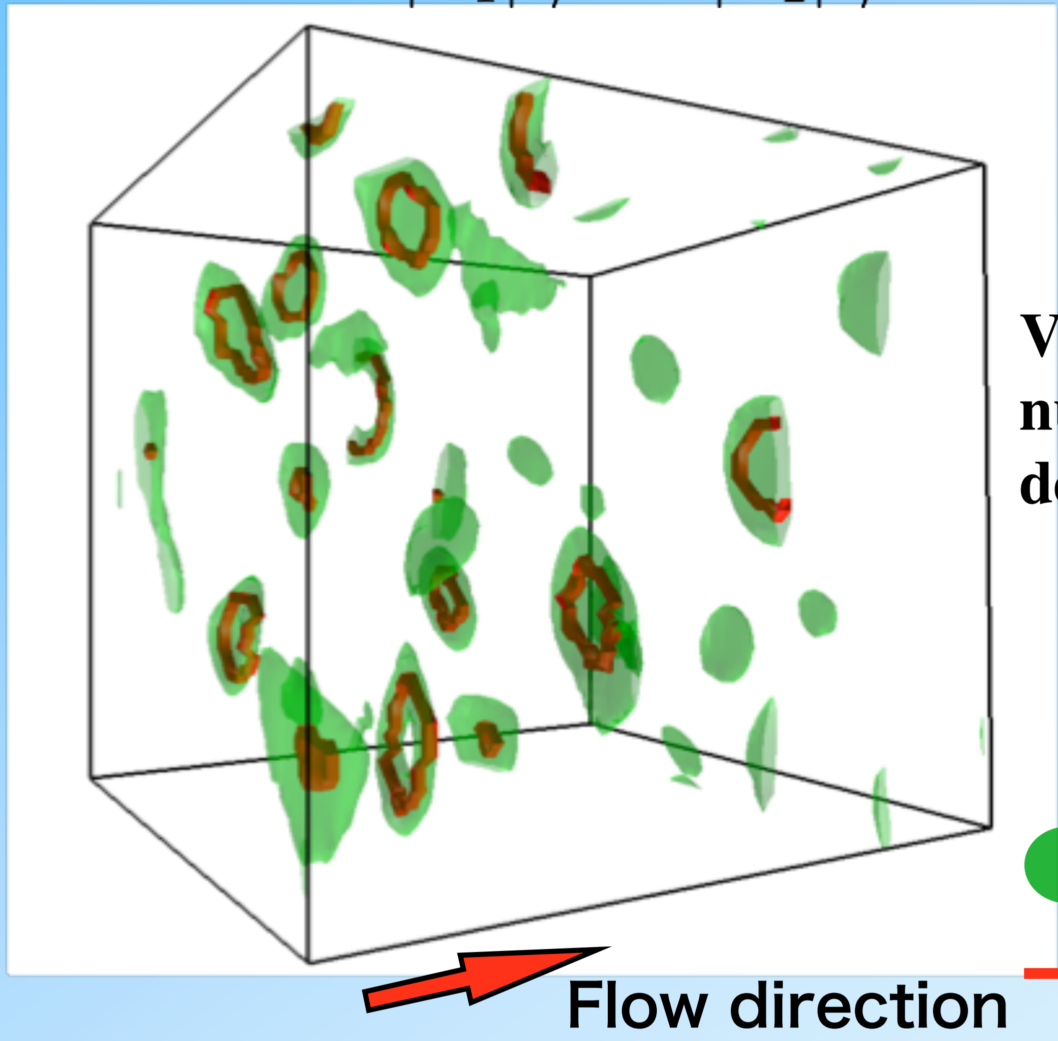


The unstable mode is amplified to lead to the disk-shaped low density regions.

Isosurface of $|\Psi_1|^2/n = 0.1$

Scenario to turbulence (2)

$$|\Psi_1|^2/n = |\Psi_2|^2/n = 1, \;\; V_R' = 4.71, \;\; \gamma = 0.9$$



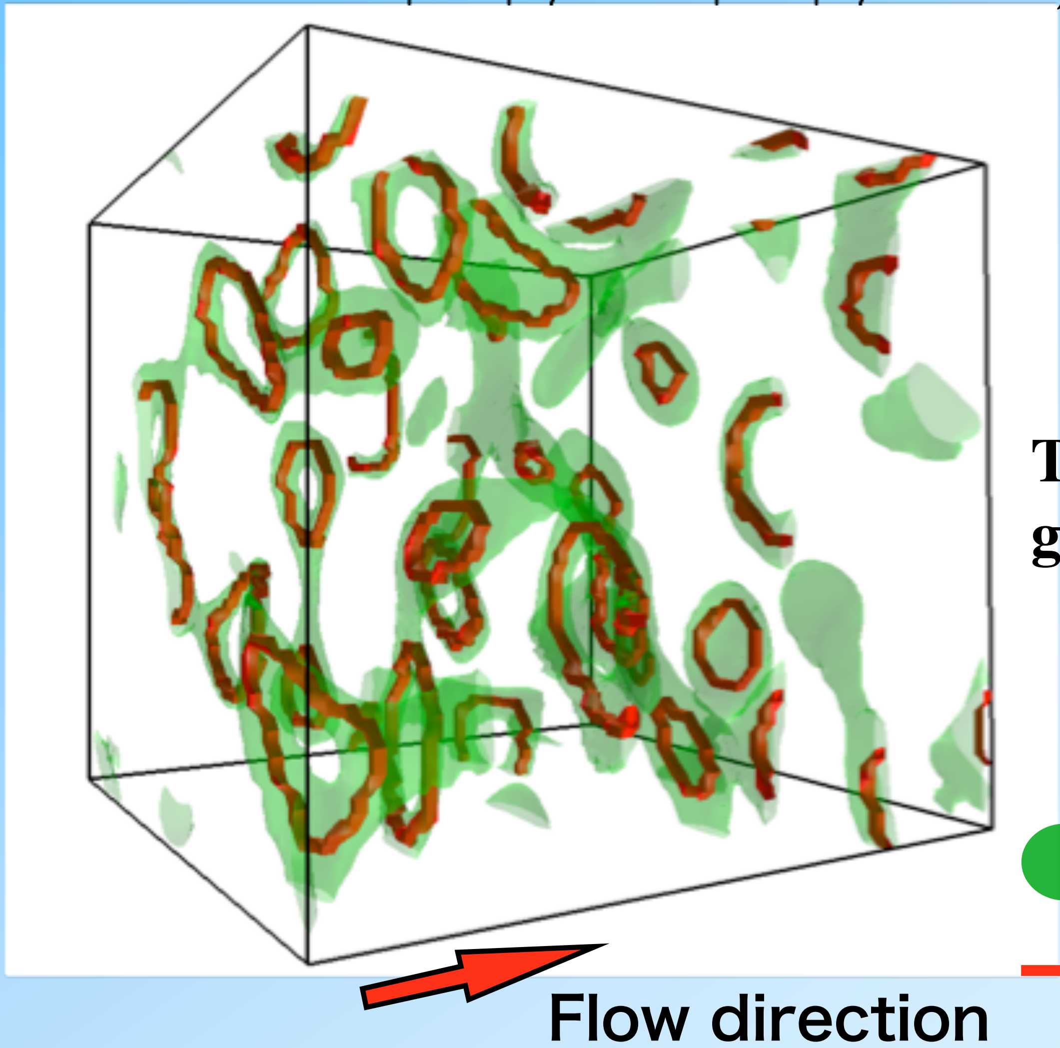
Vortex rings are nucleated inside the low density regions.

Isosurface of $|\Psi_1|^2/n = 0.1$

Vortex core of component 1

Scenario to turbulence (3)

$$|\Psi_1|^2/n = |\Psi_2|^2/n = 1,\ \ V_R' = 4.71,\ \ \gamma = 0.9$$



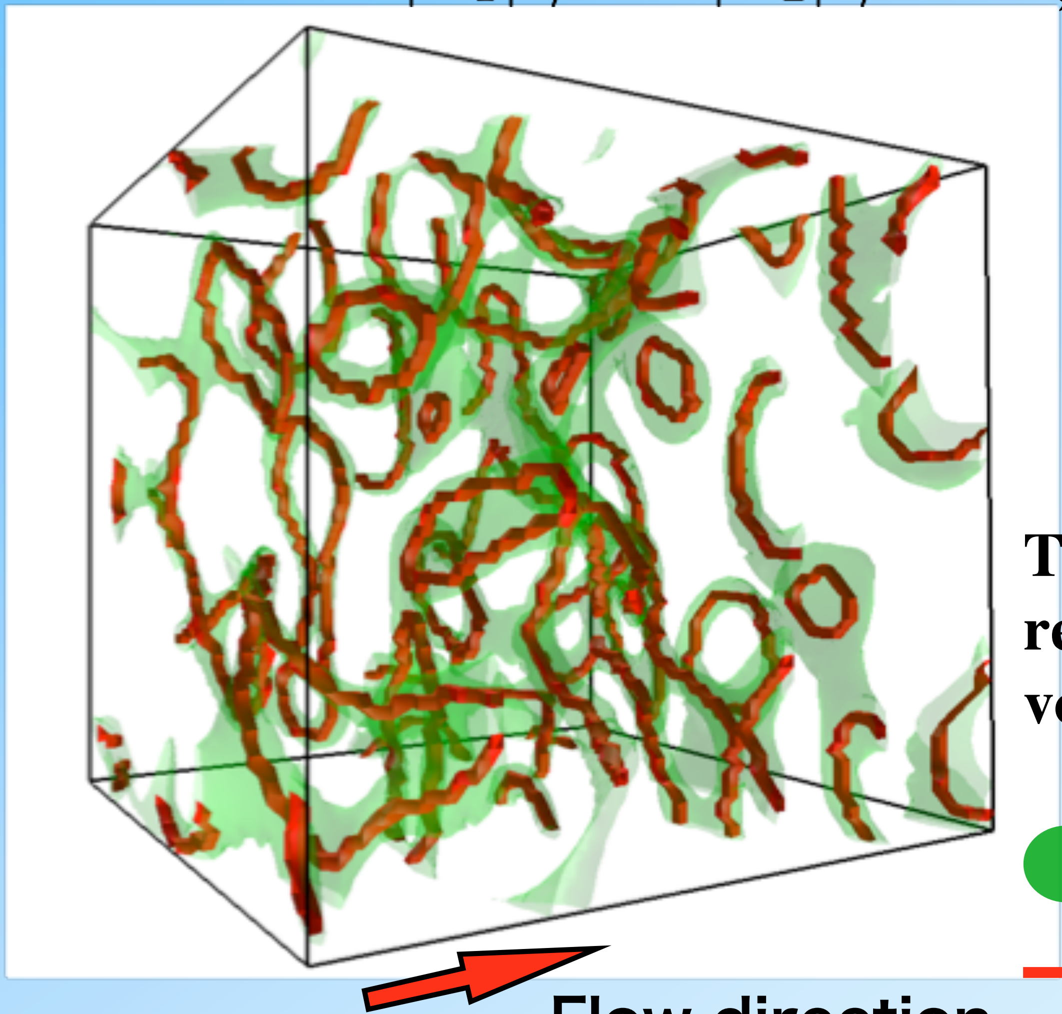
The vortices expand and grow.

Isosurface of $|\Psi_1|^2/n = 0.1$

Vortex core of component 1

Scenario to turbulence (4)

$$|\Psi_1|^2/n = |\Psi_2|^2/n = 1, \ \ V'_R = 4.71, \ \ \gamma = 0.9$$



Flow direction

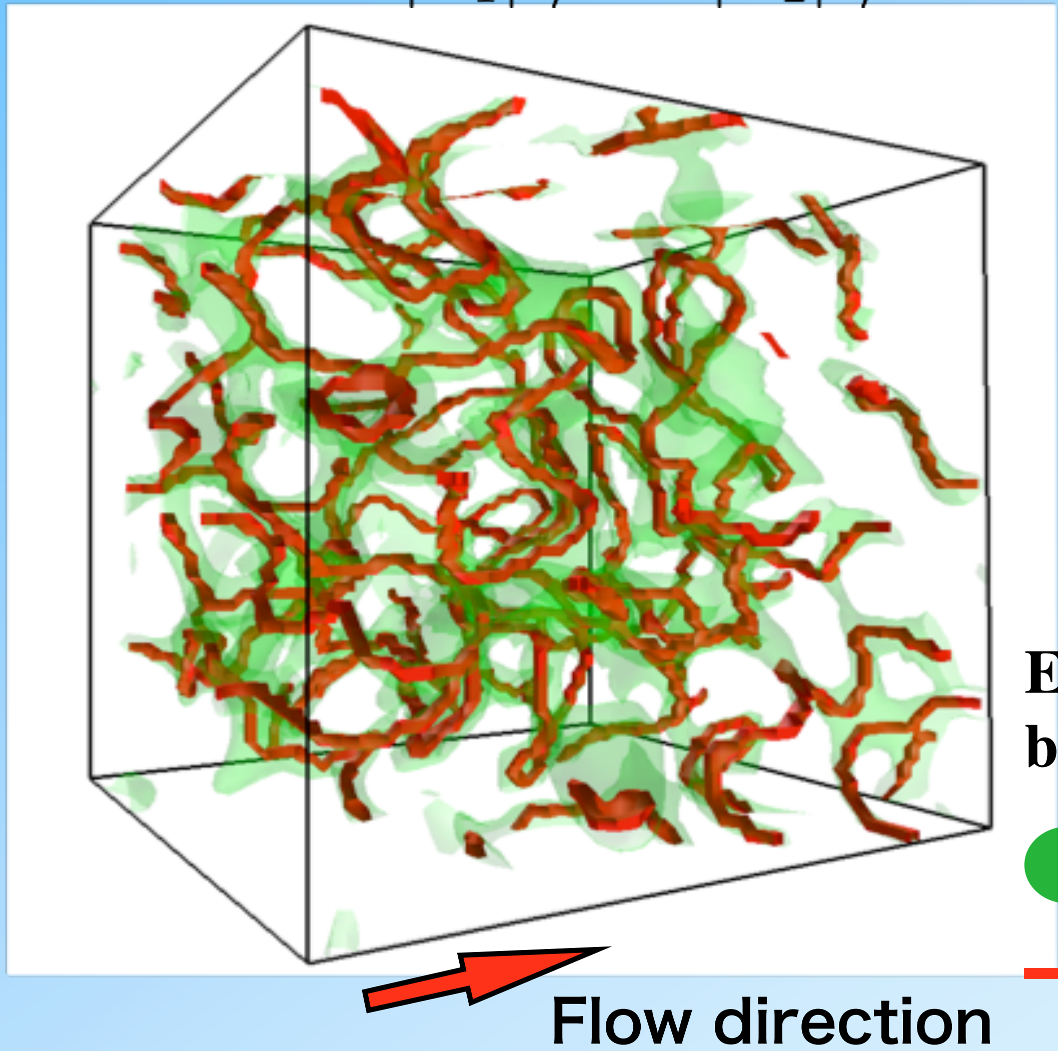
The vortices expand to reconnect with other vortices.

Isosurface of $|\Psi_1|^2/n = 0.1$

Vortex core of component 1

Scenario to turbulence (5)

$$|\Psi_1|^2/n = |\Psi_2|^2/n = 1, \;\; V_R' = 4.71, \;\; \gamma = 0.9$$

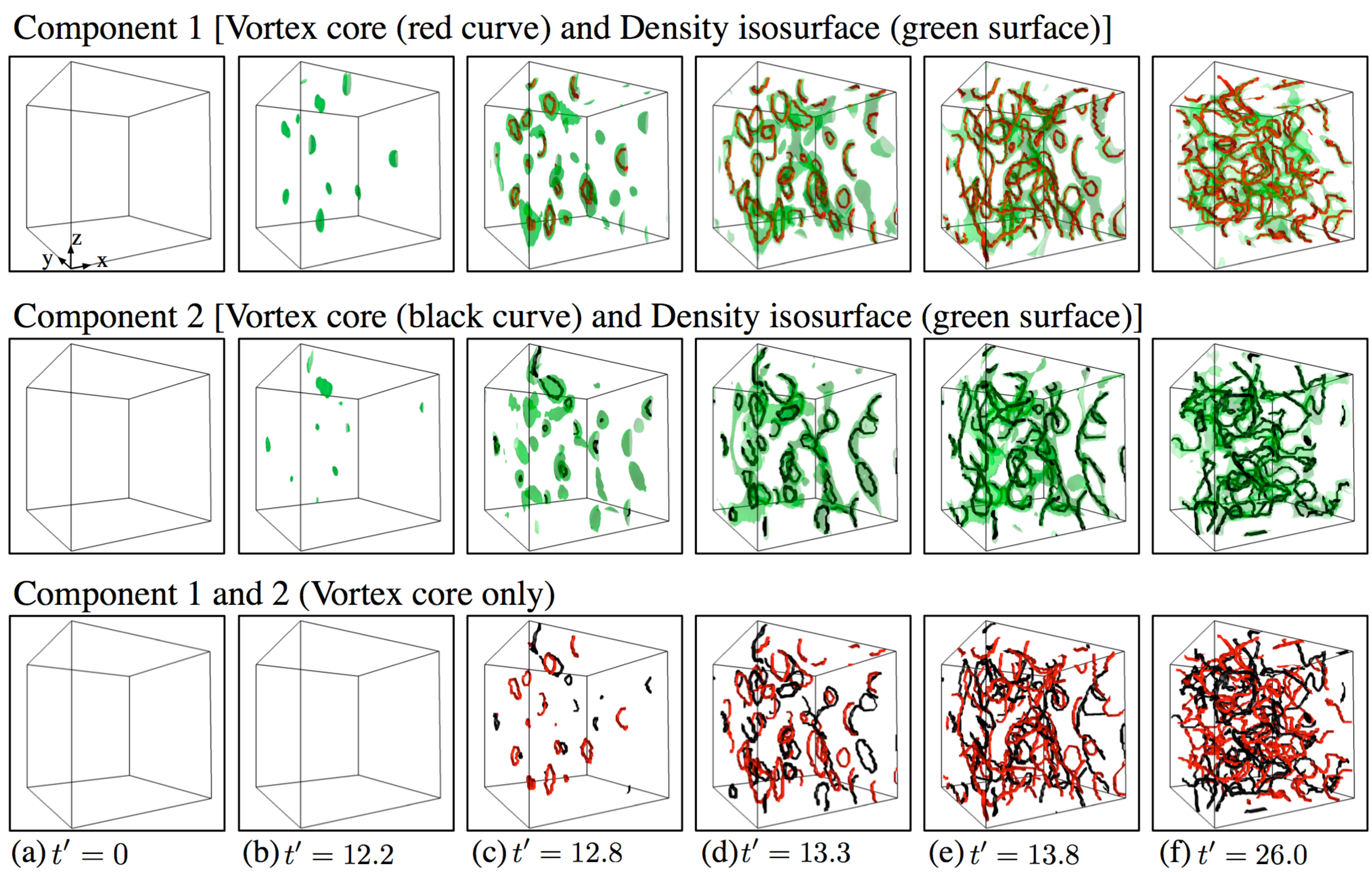


Eventually the vortices become tangled.

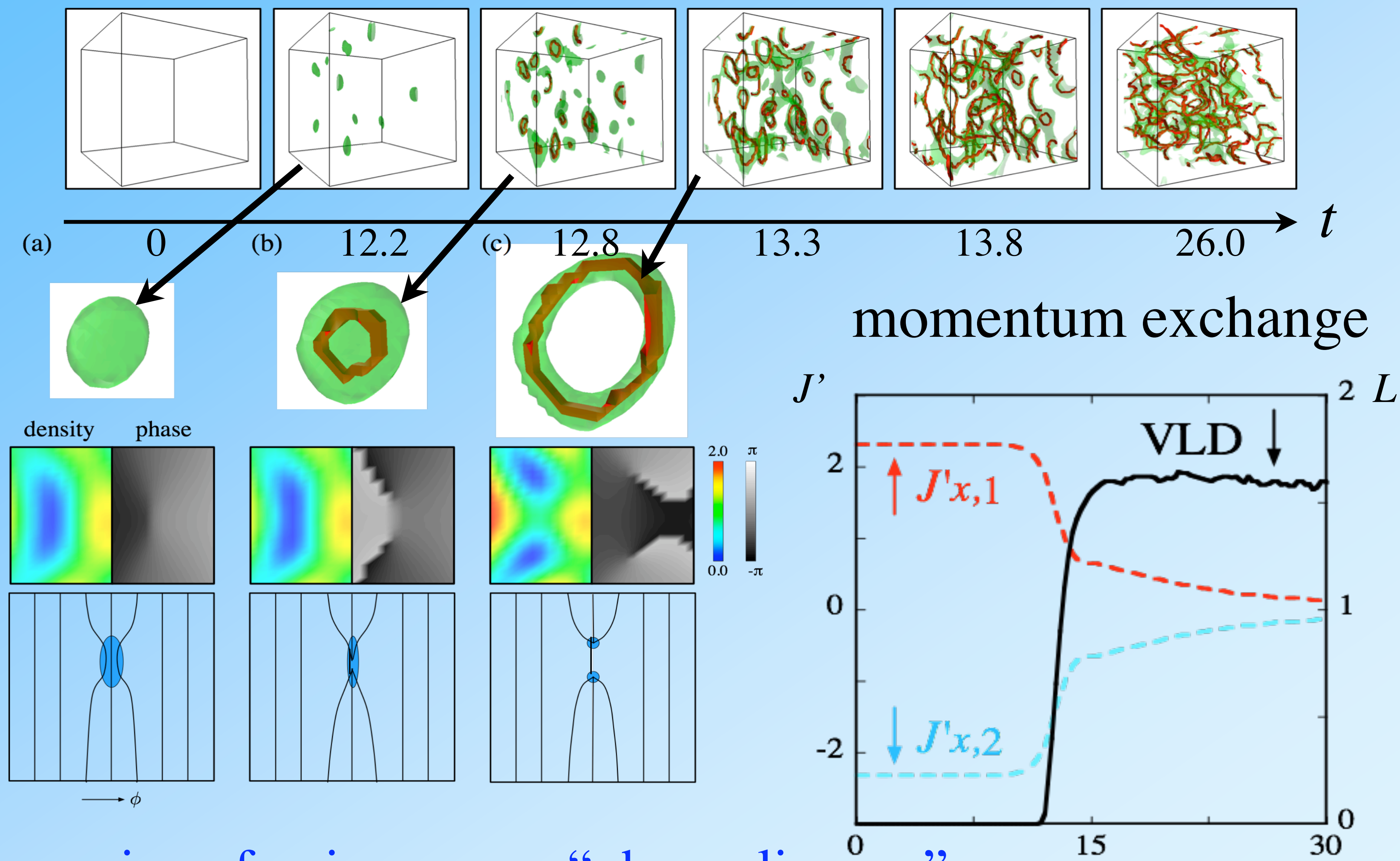
Isosurface of $|\Psi_1|^2/n = 0.1$

Vortex core of component 1

Scenario to binary quantum turbulence



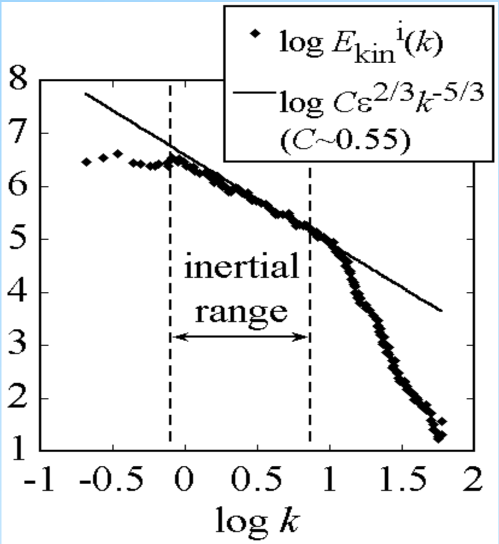
Scenario to turbulence



Expansion of a ring means "phase slippage".

Why is binary QT interesting?

We know one-component QT obeys the Kolmogorov law(K41).



M. Kobayashi, MT, J. Phys. Soc. Jpn. 74, 3248 (2005)

What happens to two-component QT?

- By changing g_{12}, we can control their coupling.
- By considering the unsymmetric case $g_{11} \neq g_{22}$, we can consider the coupling of different QTs. *etc*.

Summary

0. Introduction

   Basics of Quantum Hydrodynamics of the GP model,
   Brief research history of QT

1. Vortex lattice formation in a rotating BEC
2. QT by the GP model -Energy spectrum-
3. QT in atomic BECs
4. Quantized vortices in two-component BECs

   Quantum Kelvin-Helmholtz instability, QT